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摘要

於二度空間裡，對於前進在斜坡底床上的週期性規則表面重力波的問題，本文以Lagrangian描述方法來對其做整體的解析。將波浪場中各物理量對波浪尖銳度與底床坡度進行雙參數攝動級數展開，且將所有的控制方程式演化成系統化的模式，並由之解析出波動流場至第三階解，本文所得之三階近似解滿足自由表面常壓條件及各垂直斷面之質量通量守衡。由此三階解可以描述在斜坡底床上波動流場中由深水至臨近碎波點時之波形變化與水粒子運動軌跡變化，最後粒子運動軌跡實驗與理論比對驗證可知本文之理論結果與試驗數據相當符合。

關鍵字：Lagrangian、斜坡底床、粒子軌跡、碎波
ABSTRACT

A third-order asymptotic solution in Lagrangian description for nonlinear water wave propagating over a sloping beach is derived. The particle trajectories are obtained as a function of the nonlinear ordering parameter $\varepsilon$ and the bottom slope $\alpha$ to the third order of perturbation. A new relationship between the wave velocity and the motions of particles at the free surface profile in the waves propagating on the sloping bottom is also determined directly in the complete Lagrangian framework. This solution enables the description of wave shoaling in the direction of wave propagation from deep to shallow water, as well as the successive deformation of wave profiles and water particle trajectories prior to breaking. A series of experiment are conducted to investigate the particle trajectories of nonlinear water wave propagating over a sloping bottom. It is shown that the present third-order asymptotic solution agrees very well with the experiments.

KEYWORDS: Lagrangian, sloping bottom, particle trajectory, wave breaking.
1. INTRODUCTION

In the process of a wave propagating from deep to shallow water, the wave will deform and eventually break. Wave changes in height and its profile becomes asymmetrical during the process of shoaling. In this connection, many researchers have paid much attention to solving the wave transformation on sloping bottoms. Moreover it’s also very important at the tsunami issue by H. Segur(2007) and Constantin and Johnson(2008). However, since the sloping bottom was approximated by a large number of steps, the effects of the bottom slope could not be fully explained in many models. Biesel (1952) suggested a plausible approximation method to account for the normal incident waves propagating on a sloping plane where the bottom slope was first considered in the velocity potential as a perturbation parameter. Chen and Tang (1992) modified Biesel’s (1952) theoretical model and obtained a linear solution to the first order of the bottom slope. For nonlinear waves on the beach, Carrier and Greenspan (1958) gave an analytical solution for the shallow water wave motion of finite-amplitude, non-breaking waves on a beach of constant slope. Chu and Mei (1970) and Liu and Dingemans (1989) presented perturbation solutions for weakly nonlinear waves propagating over an uneven bottom. Chen et al. (2006) derived a fourth-order asymptotic solution of nonlinear water waves propagating normally toward a mild beach. Chen et al. (2006) used the transformation from Eulerian to Lagrangian coordinates to calculate the water particle motion up to the second order, by which the profile of a shoaling wave sequence till the breaking point can be evaluated. However, a straightforward expansion of the Eulerian solution of Stokes wave up to the third order cannot be transformed into the corresponding Lagrangian solution. Chen and Hsu (2009) presented a modified Euler–Lagrange transformation method to obtain the third-order trajectory solution in a Lagrangian form for the water particles in nonlinear water waves. Unlike an Eulerian surface, which is given as an implicit function, a Lagrangian form is expressed through a parametric representation of particle motion. Hence, the Lagrangian description is more appropriate for the free surface motion, whereas this unique feature cannot be represented by the classical Eulerian solutions (Biesel 1952, Naciri and Mei 1993, Ng 2004a–c, Chen et al. 2005, 2006, Zhang and Ng 2006, Buldakov et al. 2006, and Ng and Zhang 2007).

The first water wave theory in Lagrangian coordinates was obtained by Gerstner (1802) who assumed the flow possesses non-constant vorticity in infinite depth. Miche (1944) proposed a perturbation method for Lagrangian solution to a second order for a gravity wave motion. Pierson (1962) also applied perturbation expansion to water wave problems with Lagrangian formulae and obtained a first-order Lagrangian solution. Buldakov et al. (2006) developed a Lagrangian asymptotic formulation up to the fifth order for nonlinear water waves in deep water. In the recent for the traveling waves in irrotational flow over a flat bed, the general features of the particle paths have been obtained without the assumption of small amplitude (necessary for a power-series approach) by Contantin (2006); the particle trajectories in solitary water waves have also been gotten by Constanttin and Escher (2007); and Constantin and Strauss(2010) have extended to describe the pressure beneath a Stokes wave. Additionally, Contantion and Escher(2011) have
further exposed the analyticity of periodic traveling free surface water waves with vorticity. Chen and Hsu (2009) obtained a third-order solution for irrotational finite amplitude standing waves in Lagrangian coordinates. The particle path is similar to M. Ehrnström and E. Wahlen (2008). Hsu et al. (2010) derived a Lagrangian asymptotic solution up to the second order for short-crested waves. Asymptotic solutions up to the fifth order which describe irrotational finite amplitude progressive gravity water waves were recently derived in Lagrangian description by Chen et al. (2010). All the theories mentioned above are limited to the condition of uniform water depth. To date, only a few analytic solutions have been derived for wave transformation on a planar beach in Lagrangian coordinates. Among them, Sanderson (1985) obtained a second-order solution in a uniformly stratified fluid with a small bottom slope in a Lagrangian system. Constantin (2001) considered the Lagrangian solution for edge waves on a sloping beach. Chen and Hwung (2000) derived a linear Lagrangian solution in terms of beach slope $\alpha$ to the second order for a progressive wave propagating over a gentle plane slope, while Kapinski (2006) studied the run-up of a long wave over a uniform sloping bottom in Lagrangian description.

The purpose of this paper is to develop a nonlinear solution for surface waves propagating over a sloping bottom in a Lagrangian description and to compare the theory with a series of experiments. In order to examine the effect of a sloping bottom and wave steepness on surface waves, a perturbation expansion is used to derive an expression for the particle trajectories in terms of wave steepness $\varepsilon$ and the bottom slope $\alpha$ to the third power. The asymptotic solutions for physical quantities related to the wave motion are then obtained up to the third order. Finally, to validate the accuracy of the analytical results, a series of laboratory experiments are performed. The Lagrangian properties of particle trajectories are shown to agree with the experimental data very well.

FORMULATION OF THE PROBLEM

Consider a two-dimensional monochromatic wave propagating on a uniform gentle slope without refraction as shown in Fig. 1. The negative $x$-axis is outward to the sea from the still water level (SWL) shoreline, while the $y$-axis is taken positive vertically upward from the SWL, and the sea bottom is at $y = -d = \alpha x_0$, in which $\alpha$ denotes the bottom slope.

The fluid motion in the Lagrangian representation is described by keeping track of individual fluid particles. For two-dimensional flow, a fluid particle is identified by the horizontal and vertical parameters $(x_0, y_0)$ known as Lagrangian labels. Then fluid motion is described by a set of trajectories $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$, where $x$ and $y$ are the Cartesian coordinates. The dependent variables $x$ and $y$ denote the position of any particle at time $t$ and are functions of the independent variables $x_0$, $y_0$ and $t$. In a system of Lagrangian description, the governing equations and boundary conditions for two-dimensional irrotational free-surface flow are summarized as follows:

\[ J = \frac{\partial (x, y)}{\partial (x_0, y_0)} = x_0 y_y - x_y y_0 = 1, \]  

(1)
In Eqs. (1)–(5), subscripts \(x\), \(y\), and \(t\) denote partial differentiation with respect to the specified variable, \(P(x_0,y_0,t)\) is water pressure, \(\phi(x_0,y_0,t)\) a velocity potential function in Lagrangian system. Except for Eqs. (4) and (5) by Chen(1994), the fundamental physical relationships defining the equations above have been derived previously (e.g., by Lamb 1932; Miche 1944; Pierson 1962; Yakubovich and Zenkovich 2001). Eq. (1) is the continuity equation that set the invariant condition on the volume of a Lagrangian particle and \(y_0 = 0\) is the vertical label marked for a particle at free surface; Eq. (2) is the differentiation of Eq. (1) with respect to time. Eqs. (3) and (4) govern the irrotational flow condition and define the corresponding Lagrangian velocity potential, respectively. Eq. (5) is the Bernoulli equation for irrotational flow in Lagrangian description.

The wave motion has to satisfy a number of boundary conditions at the bottom and on the free water surface:

1. On an immovable and impermeable sloping plane with an inclination to the horizon, the no-flux bottom boundary condition gives

\[
y_t - ax_t = 0, \quad y = y_0 = -d = ax_0
\]

2. The dynamic boundary condition of zero pressure at the free surface is

\[
P = 0, \quad y_0 = 0
\]

3. A time-averaged and stationary mass flux conservation condition is required: as waves propagate toward the beach, a horizontal hydrostatic pressure gradient to balance the radiation stress of the progressive wave will produce a return flow and a boundary condition should be imposed. The additional condition usually employed is the condition of time-averaged mass flux conservation. This condition is necessary for the uniqueness of the solution and requires that at any cross-section of the \(x\)-\(y\) plane, the time-averaged mass flux should vanish (Chen et al.2003, I, 2006)

\[
y \text{ direction: } \frac{1}{T} \int_0^T \int_{-d}^0 vdy_0 dt = \frac{1}{T} \int_0^T \int_{-d}^0 v dy_0 dt = 0
\]

\[
x \text{ direction: } \frac{1}{T} \int_0^T \int_{-d}^0 udy_0 dt = \frac{1}{T} \int_0^T \int_{-d}^0 x dy_0 dt - \frac{U(\alpha)}{T} \int_0^T \int_{-d}^0 x dy_0 dt
\]
Both the superscript \( c \) and the subscript 0 express the physical quantity at \( x \to -\infty \). Because of the nonlinear effect, waves over constant depth induce a net flux of water. Thus, a constant depth streaming term is introduced in (9) which is adjusted by a unit function \( U(\alpha) \) to ensure that it can be reduced to the constant depth condition when the bottom slope is equal to zero.

**Experimental setup**

To acquire the behavior of water particle trajectory, a series of experimental measurements were carried out in a glass-walled wave tank, 20m×0.5 m×0.7 m, in Tainan Hydraulics Laboratory of National Cheng Kung University, Taiwan. A camera was set up in front of the glass wall about 10~11m from the wave generator to successively capture the particle motion with the water waves in the tank. Four wave gauges were setup at 2m, 3.05m, 4.05m and 7m from the wave generator. The whole experimental frame is schematically shown in Fig 1.

![Experimental frame and instruments setup](image)

**RESULT AND CONCLUSIONS**

**Wave Transformations**

As the height of a wave reaches its upper limit, the crest is fully developed as a summit which can be calculated as the spatial surface profile by a system of Lagrangian coordinates. In this approach, the new displacement components of water particle \( x \) and \( y \) to the third order approximation have been obtained as follows.

\[
x(x_0, y_0, t) = x_0 + \varepsilon^1 \alpha^0 f_{1,0} + \varepsilon^1 \alpha^1 f_{1,1} + \varepsilon^2 \alpha^0 (f_{2,0} + f_{2,0}') + \varepsilon^3 \alpha^0 f_{3,0}
\]

\[
y(x_0, y_0, t) = y_0 + \varepsilon^1 \alpha^0 g_{1,0} + \varepsilon^1 \alpha^1 g_{1,1} + \varepsilon^2 \alpha^0 g_{2,0} + \varepsilon^3 \alpha^0 g_{3,0}
\]

The surface wave profiles near the wave breaking point can be evaluated and the results are illustrated in Fig. 2. The linear (up to the order \( \varepsilon^1 \alpha^1 \)) and nonlinear solutions (up to the orders \( \varepsilon^2 \alpha^0 \) and \( \varepsilon^3 \alpha^0 \)) are implemented for comparison. This figure shows the surface wave profiles prior to breaking on different wave steepness and wave phase for bottom slopes of \( \alpha = 1/5 \) and 1/10, respectively, based on the wave breaking criterion of \( u/C_w = 1 \). It could be found that the third-order theory is consistent with the classification of wave breakers proposed by Galvin (1968). This confirms that the breaker type depends on the bottom slopes. In general, the third-order wave profiles are higher than the second-order and linear solutions for any wave steepness and bottom
slope. Moreover, the breaking point predicted by the third-order solution occurs earlier than that by the second-order and linear solutions.

**Particle Orbits**

The new Lagrangian solution for water-particle displacement developed in this study can be employed to demonstrate the validity for water particle motion. The parametric functions for the water particle at any position in Lagrangian coordinates \((x, y)\) are given in as Eqs. (10) and (11). Fig. 3 shows good agreement between the experimental data and the third-order asymptotic solution of the particle trajectories at the free surface.
Fig 2. Successive wave profiles prior to breaking plotted by linear (up to the order of $\epsilon^1\alpha^1$) and nonlinear solutions (up to the order of $\epsilon^2\alpha^0$ and up to the order of $\epsilon^3\alpha^0$) under varying wave conditions and bottom slopes. (Solid line: third-order solution, dot line: second-order solution, dashed line: the linear solution.)
Fig 3 a-e. Comparisons between the orbits of water particles obtained by the third-order solution, second-order solution and those from the experimental measurements of the PS motions on sloping bottom. (Circle point: experiment, Solid line: the third-order solution, dashed line: the second-order solution) (The wave conditions were listed in the Table 1.)

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