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中文摘要

本研究計畫在針對單輸入單輸出系統，探討設計一個 PID 控制器使得受控迴路系統穩定，並且能使其相關迴路轉移函數的 \( \infty H \) 模數(norm)小於特定值。吾首先將所要解決的 PID 設計問題轉換成閉迴路特徵多項式與一複數係數多項式集合的同時穩定化問題(simultaneous stabilization problem)，再利用適用於複數係數多項式的 Generalized Hermite-Biehler 定理，最後吾人得到具線性規劃特徵化(linear programming characterization)的所有達到設計要求的 PID 控制器的參數集合。

關鍵詞：PID 控制器設計、\( \infty H \)控制、線性規劃

Abstract

This research considers the problem of synthesizing proportional-integral-derivative (PID) controllers for which the closed-loop system is internally stable and the \( \infty H \) norm of a related transfer function is less than a prescribed level for a given single-input single-output plant. It is shown that the problem to be solved can be translated into simultaneous stabilization of the closed-loop characteristic polynomial and a family of complex polynomials. It calls for a generalization of the Hermite-Biehler Theorem applicable to complex polynomials. Then a linear programming characterization of all admissible \( \infty H \) PID controllers for a given plant is obtained.

Keywords: PID controller design; \( \infty H \) control; linear programming

1 Motivation and Objective

The PID controller structure is the most widely used in industrial applications. Its structural simplicity and sufficient ability of solving many practical control problems have greatly contributed to this wide acceptance. Many formulas for optimal PID controller designs can be found in the literature, and [1] provides an excellent review. Most of these design techniques are based on simple characterizations of process dynamics, for instance, the characterization by a first order model with deadtime. In spite of this, for plants having higher order, there is no generally accepted design method existing. The major obstacle to the design of optimal PID controllers, in any sense whatsoever, has been the difficulty in characterizing the entire set of stabilizing PID controllers. Indeed this difficulty holds for every fixed order and low order controller. The solution of this problem is a necessary first step to optimal, or at least, rational design of PID controllers based on achievable performance. This bottleneck has been overcome through recent results [2, 3] which have provided a computational characterization of all stabilizing PID controllers for a given but arbitrary plant. This characterization was based on a fundamental and new result generalizing the classical Hermite-Biehler Theorem [4] to the case of not necessarily Hurwitz real polynomials. Although based on this characterization, \( \infty H \) optimal design can be carried out by a brute force optimization search procedure [3, 5] within the stabilizing
controller parameter space, the related computation is highly time-consuming. The aim of this research project is to develop a computationally efficient procedure for carrying out $H_\infty$ PID optimal design instead of brute force search. Our investigation reveals that we need a complex version of the generalized Hermite-Biehler Theorem for solving $H_\infty$ PID optimal design problem. Such a result is then used for providing a computational characterization of all admissible PID controllers. The characterization for all admissible PID controllers involves the solution of a linear programming problem.

2 Approach

2.1 Problem Formulation

To this end, consider the standard feedback control system shown in Fig. 1.

\[ r \rightarrow C(s) \rightarrow N(s) \rightarrow D(s) \rightarrow y \]

**Figure 1**: Feedback control system.

Here $r$ is the command signal, $y$ is the output, $G(s) = \frac{N(s)}{D(s)}$ is the plant to be controlled, $N(s)$ and $D(s)$ are coprime polynomials, and $C(s)$ is the controller used for making the closed-loop system stable and achieving desired design specifications. In this research project, the controller $C(s)$ is chosen to be a PID controller, i.e.

\[ G(s) = k_p + \frac{k_i}{s} + k_ds + k_ds^2. \]  

(1)

In particular, the following three closed-loop transfer functions are considered: the sensitivity function: $S(s) = \frac{1}{1+C(s)G(s)}$, the complementary sensitivity function: $T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$, and the input sensitivity function: $U(s) = \frac{C(s)}{1+C(s)G(s)}$. Various performance and robustness specifications could be made by using the $H_\infty$-norm of weighted versions of $S(s)$, $T(s)$, and $U(s)$. Now, for $C(s)$ to be a PID controller, $S(s)$, $T(s)$, and $U(s)$ can be presented in the general form:

\[ T_\infty(s,k_p, k_i, k_d) = \frac{A(s) + (k_d s^2 + k_i s + k_p)B(s)}{sD(s) + (k_d s^2 + k_i s + k_p)N(s)} \]

for some real polynomials $A(s)$ and $B(s)$. For the transfer function $T_\infty(s, k_p, k_i, k_d)$ and a given number $\gamma_0 > 0$, define the $H_\infty$ optimization criteria to be:

\[ \|W(s)T_\infty(s, k_p, k_i, k_d)\| < \gamma_0 \]  

(2)

where $W(s)$ is a stable frequency-dependent weighting function that is selected to represent the particular stability and performance specifications relevant to the desired design objective at hand.

The objective of this research project is to solve the problem of finding PID controllers, if any, for which the closed-loop system is internally stable and $\|W(s)T_\infty(s, k_p, k_i, k_d)\| < \gamma_0$ for a given positive number $\gamma_0$.

Now let the weighting function $W(s) = \frac{W_w(s)}{W_d(s)}$, where $W_w(s)$ and $W_d(s)$ are coprime polynomials; moreover, $W_d(s)$ is Hurwitz. For notational simplicity, we write the closed-loop characteristic polynomial

\[ \rho(s, k_p, k_i, k_d) = sD(s) + (k_d s^2 + k_i s + k_p)N(s) \]

and

\[ \varphi(s, k_p, k_i, k_d, \gamma_0, \theta) = |sW_w(s)D(s) + \frac{1}{\gamma_0} e^{\gamma_0 W_w(s)A(s)}| + (k_d s^2 + k_i s + k_p) \times |sW_w(s)N(s) + \frac{1}{\gamma_0} e^{\gamma_0 W_w(s)B(s)}|. \]

Based on Lemma 9.2 of [6], we are already in the position to recast synthesis of $H_\infty$ PID controllers into the simultaneous polynomial stabilization problem: Given a weighted closed-loop transfer function of the form

\[ W(s)T_\infty(s, k_p, k_i, k_d) = \frac{W_w(s)A(s) + (k_d s^2 + k_i s + k_p)B(s)}{sD(s) + (k_d s^2 + k_i s + k_p)N(s)} \]
Let \( \rho(s,k_p,k_i,k_d) \) and \( \varphi(s,k_p,k_i,k_d,\gamma,\theta) \) be already defined. For a given \( \gamma_0>0 \), there exist PID gain values \( (k_p, k_i, k_d) \) such that \( W(s)T_r(s, k_p, k_i, k_d) < \gamma_0 \) if and only if the following three conditions hold:

1. \( \rho(s,k_p,k_i,k_d) \) is Hurwitz;
2. \( \varphi(s,k_p,k_i,k_d,\gamma,\theta) \) is Hurwitz for all \( \theta \) in \([0, 2\pi)\);
3. \( W(\infty)T_r(\infty, k_p, k_i, k_d) < \gamma_0 \).

2.2 An Extension of PID Stabilization to the Case of Complex Polynomials

Now we consider a complex polynomial of the form:

\[
\delta(s,k_p,k_i,k_d) = L(s) + (k_d s^2 + k_p s + k_i) M(s)
\]

where \( L(s) \) and \( M(s) \) are two arbitrary complex polynomials. In this subsection, we will focus on the problem of determining those real values of \( (k_p, k_i, k_d) \), if any, for which (3) is Hurwitz stable. Note that by setting \( L(s) = sD(s) \) and \( M(s) = N(s) \), the problem stated above becomes PID stabilization. We, therefore, refer to the extension of the PID stabilization problem. In this subsection, we will provide the complete characterization of the set of all admissible values of \( (k_p, k_i, k_d) \).

To this end, we consider

\[
L(s) = (a_0 + j b_0) + (a_1 + j b_1)s + \cdots + (a_m + j b_m)s^m
\]

\[
M(s) = (c_0 + j d_0) + (c_1 + j d_1)s + \cdots + (c_m + j d_m)s^m
\]

and the following real-imaginary decompositions of \( L(s) \) and \( M(s) \):

\[
L(s) = L_R(s) + L_I(s)
\]

\[
M(s) = M_R(s) + M_I(s)
\]

where

\[
L_R(s) = a_0 + j b_0 + a_2 s^2 + j b_2 s^2 + \cdots
\]

\[
L_I(s) = j b_0 + a_1 s + j b_1 s + a_3 s + \cdots
\]

\[
M_R(s) = c_0 + j d_0 + c_2 s^2 + j d_2 s^2 + \cdots
\]

\[
M_I(s) = j d_0 + c_1 s + j d_1 s + c_3 s + \cdots
\]

Define

\[
M^*(s) = M_R(s) - M_I(s).
\]

Also let \( n, m \) be the degrees of \( \delta(s,k_p,k_i,k_d) \) and \( M(s) \) respectively. Then we obtain

\[
\delta(j \omega, k_p, k_i, k_d) M^*(j \omega) = p(\omega, k_p, k_i, k_d) + j q(\omega, k_p, k_i, k_d)
\]

where

\[
p(\omega, k_p, k_i, k_d) = p_1(\omega) + (k_d - k_p \omega^2) p_2(\omega)
\]

\[
q(\omega, k_p, k_i, k_d) = q_1(\omega) + k_i q_2(\omega)
\]

\[
p_1(\omega) = L_q(j \omega) M^*(j \omega) - L_r(j \omega) M^*(j \omega)
\]

\[
p_2(\omega) = M^*_q(j \omega) - M_r(j \omega)
\]

\[
q_1(\omega) = \frac{1}{j} [L_r(j \omega) M^*_q(j \omega) - L_q(j \omega) M_r(j \omega)]
\]

\[
q_2(\omega) = \omega [M^*_q(j \omega) - M_r(j \omega)]
\]

Also, define \( p_j(\omega, k_p, k_i, k_d) = \frac{p(\omega, k_p, k_i, k_d)}{(1 + \omega^2)^{\frac{m-1}{2}}} \) and

\[
q_j(\omega, k_p, k_d) = \frac{q(\omega, k_p, k_i, k_d)}{(1 + \omega^2)^{\frac{m+1}{2}}}
\]

The formal statement of our main result on the extension of the PID stabilization problem requires the introduction of the following definitions.

**Definition 2.2.1** Let \( m, n \), \( q_j(\omega, k_p, k_d) \) be as already defined. Denote \( \xi \) to be the leading coefficient of \( \delta(s,k_p,k_i,k_d) M^*(s) \). For a given fixed \( k_p \), let \( \omega_1 < \omega_2 < \cdots \omega_{i-1} \) be the real, distinct finite zeros of \( q_j(\omega, k_p, k_d) \) with odd multiplicities. Also define \( \omega_0 = -\infty \) and \( \omega_i = +\infty \). Define a sequence of numbers \( \omega_0, \omega_1, \omega_2, \ldots, \omega_i \) as follows:

\[
A_{\xi} = \begin{cases} \{i_0, i_1, \ldots, i_t\} & \text{if } m + n \text{ is even and } \xi \text{ is purely real, or } m + n \text{ is odd and } \xi \text{ is purely imaginary} \\ \{i_1, i_2, \ldots, i_{t-1}\} & \text{if } m + n \text{ is even and } \xi \text{ is not purely real, or } m + n \text{ is odd and } \xi \text{ is not purely imaginary} \end{cases}
\]

where

\[
A_{\xi} = \{1, 1\} \text{ if } j \omega_0 \text{ is not a } j \omega \text{ axis root of } M^*(s).
\]

\[
A_{\xi} = \{0\} \text{ otherwise.}
\]

**Definition 2.2.2** Let \( m, n, \xi, q_j(\omega, k_p) \), \( q_j(\omega, k_p) \) be as already defined. For a given fixed \( k_p \), let \( \omega_1 < \omega_2 < \cdots < \omega_{i-1} \) be the real, distinct finite zeros of \( q_j(\omega, k_p) \) with odd
multiplicities. Also define $\omega_0 = -\infty$ and $\omega_i = +\infty$. For each string $I = \{i_0, i_1, \ldots, i_l\}$ or $\{i_1, i_2, \ldots, i_{l-1}\}$ in $A_{k_p}$, let $\gamma(I)$ denote the signature associated with the string $I$ defined by

$$
\gamma(t) = \left\{ \begin{array}{ll}
\frac{1}{2} \left\{ i_0 \cdot (-1)^{-i_0} + 2 \sum_{i=1}^{l} i_i \cdot (-1)^{-i} \right\} \sgn \left( \begin{array}{c} \omega_0, k_p \end{array} \right) & \text{if } m + n \text{ is even and } \xi \text{ is purely real, or if } m + n \text{ is odd and } \xi \text{ is purely imaginary.}

\end{array} \right.
$$

(4)

Definition 2.2.3 The set of strings in $A_{k_p}$ with a prescribed signature $\gamma = \phi$ is denoted by $A_{k_p}(\phi)$. For a given fixed $k_p$, we also define the set of admissible strings for the extended PID stabilization problem as

$$
F_{k_p}^* = A_{k_p}(n + \sigma(M^*(s))).
$$

We are now ready to state the main result.

Theorem 2.2.1 (Main Result on Extended PID Stabilization) The extended PID stabilization problem, with a fixed $k_p$, is solvable for given complex polynomials $L(s)$ and $M(s)$ if and only if the following conditions hold:

(i) $F^*_k$ is not empty where $F^*_k$ is as already defined, i.e., at least one feasible string exists and

(ii) There exists a string $I = \{i_0, i_1, \ldots, i_l\}$ or $\{i_1, i_2, \ldots, i_{l-1}\} \in F^*_k$ and values of $k_i$ and $k_d$ such that $\forall t = 0, 1, \ldots, l$ or $1, 2, \ldots, l - 1$ for which $M^*(j \omega) = 0$

$$
p(\omega, k, k_d) > 0. \quad (5)
$$

where $p(\omega, k, k_d)$ is as already defined. Furthermore, if there exist values of $k_i$ and $k_d$ such that the above condition is satisfied for the feasible strings $I, I_2, \ldots, I_l \in F^*_k$, then the set of stabilizing $(k_i, k_d)$ values corresponding to the fixed $k_p$ is the union of the $(k_i, k_d)$ values satisfying (5) for $I, I_2, \ldots, I_l$.

Proof: The proof is based on the complex version of the generalized Hermite–Biehler Theorem [7]. It is very similar to that given in [3] for PID stabilization and is, therefore, omitted.

3 Results on $H_{\infty}$ Control Design Using PID Controllers

In Subsection 2.1, we showed that the problem of synthesizing $H_{\infty}$ PID controllers can be formulated in terms of three conditions involving simultaneous polynomial stabilization. In this section, we can now use the results of the last section to solve the stabilization problem in question. First using Theorem 2.2.1, for any fixed $k_p$ and by setting $L(s) = sD(s)$ and $M(s) = N(s)$, we can solve a linear programming problem to determine the stabilizing set of $(k_i, k_d)$ values such that $\rho(s, k_p, k_i, k_d)$ is Hurwitz. Let this set be denoted by $S_{(k, k_i, k_d)}$. With such a fixed $k_p$, any fixed $\theta \in [0, 2\pi)$ and by setting $L(s) = sW_0(s)D(s) + \frac{1}{T_0}e^{s\omega\theta}W_0(s)A(s)$ and $M(s) = W_0(s)N(s) + \frac{1}{T_0}e^{s\omega\theta}W_0(s)B(s)$ and using Theorem 2.2.1, again we can solve a linear programming problem to determine the admissible set of $(k_i, k_d)$ values for which $\phi(s, k_p, k_i, k_d, \gamma, \theta)$ is Hurwitz. Let this set be denoted by $S_{(k, k_i, k_d)}$. By keeping $k_p$ fixed and letting $\theta^*$ vary in $[0, 2\pi)$, we can determine the set of admissible $(k_i, k_d)$ values for which the entire $\phi(s, k_p, k_i, k_d, \gamma, \theta)$ is Hurwitz. This set is denoted by $S_{(2, k_i, k_d)}$ and is defined as

$$
S_{(2, k_i, k_d)} = \cap_{\theta \in [0, 2\pi)} S_{(2, k_i, k_d, \theta)}.
$$

Again with the same fixed $k_p$, we denote $S_{(3, k, k_i)}$ to be the admissible set of $(k_i, k_d)$ values for which $|W(\omega)\tau_0(\omega, k_p, k_i, k_d)| < \gamma_\omega$. Once again, the admissible solutions to $|W(\omega)\tau_0(\omega, k_p, k_i, k_d)| < \gamma_\omega$ can be obtained by solving a linear programming problem. Now for a fixed $k_p$, the set of all admissible $(k_i, k_d)$ values for which
\[ \| w(s)T(s, k_p, k_i, k_d) \| < \gamma_s \] denoted by \( S_{(\gamma_s)} \) is given by
\[ S_{(\gamma_s)} = \cap_{i=1, 2, \ldots} S_{(\gamma_s, \theta_i)}. \]
The set of all admissible \((k_p, k_i, k_d)\) values such that \( \| w(s)T(s, k_p, k_i, k_d) \| < \gamma_s \) can now be found by simply sweeping over \( k_p \) and determining \( S_{(\gamma_s, \theta_i)} \) at each stage. The necessary stabilizing range of \( k_p \) values over which the sweeping has to be carried out can be \textit{a priori} narrowed down by using the root locus ideas presented in [3].

**Remark 3.1** Note that from Theorem 2.2.1, since the constraint set is linear, the admissible set for (5) is either a convex polygon or an intersection of half planes. Also it can be easily shown that \( S_{(\gamma_s, \theta_i)} \) is a union of convex sets. Therefore, for a fixed \( k_p \), the admissible \((k_i, k_d)\) gain values such that \( \| w(s)T(s, k_p, k_i, k_d) \| < \gamma_s \) is a union of convex sets.

**Remark 3.2** For a fixed \( k_p \), \( H_\infty \) PID controller design in [3, 5] was carried out by a brute force search procedure sweeping within the stabilizing \((k_i, k_d)\) plane. Instead, the design procedure presented here involves the solution of a one-parameter family of linear programming problems. Thus, the proposed design procedure is much more computationally efficient.

We now present a simple example to illustrate the detailed calculations involved in \( H_\infty \) PID controller design.

**Example 3.1** Consider the plant \( G(s) = \frac{N(s)}{D(s)} \) where
\[ N(s) = s - 1 \]
\[ D(s) = s^3 + 0.8s - 0.2 \]
and the PID controller
\[ C(s) = \frac{k_p s^2 + k_i s + k_d}{s}. \]
In this example, we consider the problem of determining the admissible PID controller gain values for which \( \| w(s)T(s) \| < 1 \), where \( T(s) \) is the complementary sensitivity function:
\[ T(s) = \frac{(k_p s^2 + k_i s + k_d)(s - 1)}{s^2(s^2 + 0.8s - 0.2) + (k_p s^2 + k_i s + k_d)(s - 1)} \]
and the weight \( w(s) \) is chosen as a high pass transfer function:
\[ w(s) = \frac{s + 0.1}{s + 1}. \]

From Subsection 2.1, we know that the admissible \((k_p, k_i, k_d)\) values exist if and only if the following conditions hold:
1. \( \rho(s, k_p, k_i, k_d) = s + (s^2 + 0.8s - 0.2) + (k_p s^2 + k_i s + k_d)(s + 1) \) is Hurwitz;
2. \( \phi(s, k_p, k_i, k_d, \theta) = s(s+1)(s^2 + 0.8s - 0.2) + (k_p s^2 + k_i s + k_d)(s + 1) + e^{j\theta}(s + 0.1)(s + 1) \) is Hurwitz for all \( \theta \) in \([0, 2\pi]\);
3. \( \| W(\infty)T(\infty) \| = \left| \frac{k_p}{k_p + 1} \right| < 1. \)

For the condition (1), with a fixed \( k_p \), for instance \( k_p = -0.35 \), and by setting \( L(s) = s(s^2 + 0.8s - 0.2) \) and \( M(s) = s - 1 \), and using the results of Theorem 2.2.1, we obtain the admissible set \( S_{(1, \cdot, -0.35)} \) for which the closed-loop system is stable. Now fixing \( k_p = -0.35 \), setting \( L(s) = s(s^2 + 0.8s - 0.2) \) and \( M(s, \theta) = (s + 1)(s - 1) + e^{j\theta}(s + 0.1)(s + 1) \), sweeping over \( \theta \in [0, 2\pi] \) and using the results of Theorem 2.2.1 at each stage, we obtained the admissible set \( S_{(2, \cdot, -0.35)} \) which gives the constraint that \( k_d > -0.5 \). Hence the admissible set \( S_{(\cdot, 1, -0.35)} \) is given by \( S_{(\cdot, 1, -0.35)} = \{ (k_i, k_d) | k_i \in R, k_d > -0.5 \} \). Then for \( k_p = -0.35 \), the admissible set of \((k_i, k_d)\) values for which \( \| W(s)T(s) \| < 1 \) is the intersection of \( S_{(1, \cdot, -0.35)} \), \( S_{(2, \cdot, -0.35)} \), and \( S_{(\cdot, 1, -0.35)} \). The intersection is \( S_{(\cdot, 1, -0.35)} \) sketched in Fig. 2. Now, using root locus ideas [3], it was determined that a necessary condition for the existence of stabilizing \((k_i, k_d)\) values is that \( k_p \in (-1.8, -0.2) \). Thus, by sweeping over \( k_p \in (-1.8, -0.2) \), and repeating the above procedure, we obtained the admissible set of \((k_p, k_i, k_d)\) values for which \( \| W(s)T(s) \| < 1 \). The entire admissible set is sketched in Fig. 3.
**Figure 2:** The admissible set

\[ S_{(-0.35)} = \bigcap_{i=1,2,3} S_{(\epsilon, -0.35)}. \]

**Figure 3:** The admissible set of \((k_p, k_i, k_d)\) values for which \(\| W(s) T(s) \|_\infty < 1\).

### 4 Concluding Remarks

In this research report, we have provided a computational solution to the problem of characterizing all admissible PID controllers for which the closed-loop system is internally stable and the \(H_\infty\) - norm of a related transfer function is less than a given level. The characterization was based on a generalized Hermite-Biehler Theorem for complex polynomials. This characterization besides being computationally efficient reveals important structural properties of \(H_\infty\) PID controllers. It has been shown that for a fixed proportional gain, the set of admissible integral and derivative gains lie in a union of convex sets.

### 5 Self-evaluation of Significance

The results of this research are useful both from a practical as well as theoretical point of view. On the practical side, it provides an insightful characterization of all admissible \(H_\infty\) PID controllers that gives a substantial aid in practical PID controller design. On the theoretical side, it may open up the possibility of robustly optimizing various performance criteria when the controller structure is constrained by hardware considerations. The results of this research will be published in the *Proceedings of the 40th IEEE Conference on Decision and Control* [7].

### 6 References


