Transport properties of composites consisting of periodic arrays of exponentially graded cylinders with cylindrically orthotropic materials

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The work is concerned with the determination of effective conductivities and field potentials of matrix-based composites consisting of periodic arrays of cylinders which are cylindrically orthotropic and exponentially graded along the radial direction. We generalize Rayleigh’s method to account for the periodic arrangements of these cylinders. The potential field and effective conductivities of composite systems were calculated to a very high order to achieve a sufficient accuracy. We find that the cylindrical orthotropy of the inclusions has a dramatic effect on the potential field of the inclusions. In addition, we discuss the effect of the grading factor on the effective conductivity. Interestingly, we find that when the inclusions are purely cylindrically orthotropic, their effects can be fully described by homogeneous isotropic cylinders. This equivalent isotropic conductivity is simply the geometric mean of the radial and tangential conductivities of cylindrically orthotropic cylinders. © 2005 American Institute of Physics.

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I. INTRODUCTION

The determination of effective conductivity of composites is one of the classical problems in micromechanics of heterogeneous media. An earlier celebrated work dated back to more than one century ago by Lord Rayleigh,1 who proposed an ingenious method to accurately calculate the field potential and effective conductivity of composites with periodic arrangements of isotropic reinforcements. In the past developments of this subject, one branch is directed toward the derivations of upper and lower bounds for the effective tensors of composites in terms of constituent properties, volume fractions, and statistical information. The other branch is to provide estimates for the effective conductivities based on various approximate and heuristic models. For a good overview of the subject, the reader is referred to the book of Milton.2

In the literature, most of the studies considered that the constituent properties are isotropic or rectilinearly anisotropic, and that each phase possesses a constant property. However, in real systems, there are situations in which the constituent properties are curvilinearly anisotropic and non-homogeneous (i.e., spatially varying). Materials with curvilinear anisotropy possess constant properties in specific curvilinear coordinate systems. For example, cylindrical orthotropy is characterized by the fact that the conductivities in the radial, tangential, and axial directions are distinct. They appear, for example, in carbon fiber3 and in wood. On the other aspect, materials with spatially varying properties, also referred to as functionally graded materials, have found merits in quite a few engineering applications.4 For a good review of research on functionally graded materials, the reader is referred to Hirai5 and Suess.6

The present paper is concerned with the determination of the potential field and effective transport behavior of composites consisting of periodic arrays of cylinders which are cylindrically orthotropic and exponentially graded along the radial direction. The assumption of an exponentially varying property is common in the engineering literature in modeling functionally graded materials. Relevant works include those of Giannakopoulos and Suresh,7 Martin et al.,8 and the references contained therein. Physically, this simulates an inhomogeneous cylinder with a specific continuously varying conductivity along the radial direction. In this work, we have employed an analytical method, the Rayleigh’s method,1,13 to calculate the field potential and effective conductivity of the considered periodic composites. Relevant works on effective conductivities of periodic composites with isotropic cylinders are many. For example, Perrins et al.14 derived the transport properties of circular cylinders in square and hexagonal arrays. McPhedran and McKenzie15 considered the electrostatic potential problem for arrays of cylinders with zero applied field (resonant solutions). Nicorovici et al.16 studied the effective dielectric constants of a square array of coated cylinders. Yardley et al.17 and Nicorovici and McPhedran18 used the Rayleigh method to determine the effective transport properties of a rectangular array of elliptical cylinders. Parallel studies has also been carried out for spherical inclusions (see, for example, Poladian and McPhedran19 and for elasticity (see, for example, McPhedran and Movchan20). The problem of the effective transport properties of a three-phase matrix-based composite is considered by McPhedran.21 For composites reinforced by (coated) cylindrically orthotropic fibers, Benveniste et al.22 estimated the effective conductivity based on the Mori-Tanaka mean-field approach. Other works include that of Hatta and Taya23 who considered the

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transport behavior of composites with spheroidal inclusions. To our knowledge, the subject of periodic composites with cylindrically orthotropic and exponentially graded cylinders has not been examined in the literature before.

The paper is organized as follows. In Sec. II, we state the considered physical problem. Section III derives the admissible potential for cylindrically orthotropic and exponentially graded cylinders. For the considered cylinders with spatially varying properties, we find that the admissible potential can still be derived in analytic forms, although the mathematics is much more involved. We demonstrate that the Rayleigh's identity for homogeneous isotropic cylinders remains valid for the inhomogeneous cylindrically orthotropic inclusions. A simple formula, valid for the present system, was derived in Sec. IV. We have computed the complete set of expansion coefficients within sufficient accuracy. We demonstrate that the cylindrical orthotropy of the inclusions has a dramatic effect on the potential field of the inclusions. When the inclusions become isotropic, we have verified our results with the previous known solutions. The effects of cylindrical orthotropy and the grading factor are exemplified numerically. Lastly, some concluding remarks are made in Sec. VI.

II. PROBLEM STATEMENT

We consider a composite medium consisting of a periodic array of cylinders arranged in a homogeneous isotropic matrix with conductivity \( k_m \). The cylinders are cylindrically orthotropic with different conductivities in the radial and circumferential directions, denoted by \( k_r \) and \( k_\theta \), respectively. In addition the cylinders may exhibit an exponential grading along the radial direction. The objective is to assess the effects of cylindrical orthotropy and exponential grading on the effective transport properties of this periodic composite. As the intensity along the axial direction is the same for all phases, the overall conductivity in the axial direction is simply the Voigt's estimate from the constituent conductivities. Accordingly, we shall be concerned with the behavior due to a transverse intensity only.

For convenience, let us first introduce a Cartesian coordinate system \((x_1, x_2)\) positioned at the center \(O\) of one of the cylinders (Fig. 1). The sides of the rectangular cell parallel to the \( x_1 \) and \( x_2 \) coordinates are, respectively, denoted by \( a \) and \( b \), and the cylinders are of the same size with radius \( \rho \). Without loss of generality, let us apply a uniform intensity \( E \) along the positive \( x_1 \) axis, i.e., \( \varphi|_{x_2=0} = -E x_1 \), where \( \varphi \) is the temperature potential. Under steady-state conditions, the temperature field inside the unit cell \( \Omega \) is governed by

\[
\nabla \cdot [k(x) \nabla \varphi] = 0 \quad \text{in } \Omega,
\]

where

\[
k(x) = \begin{cases} k_m I & \text{if } x \in \Omega \setminus V \\ k_i & \text{if } x \in V. \end{cases}
\]

Here \( k_i \) denotes the conductivity of the inclusion which may possibly vary with \( x \), \( I \) is the identity matrix, and \( V \) is the volume occupied by the circular cylinder. At the interface \( \partial V \), we assume that the inclusion and the matrix are perfectly bonded. This means that the temperature potential and the normal component of heat flux vector are continuous across the interfaces,

\[
\varphi_{m|\partial V} = \varphi_{i|\partial V}, \quad k_m \nabla \varphi_m \cdot n|_{\partial V} = k_i \nabla \varphi_i \cdot n|_{\partial V},
\]

where \( n \) is the unit normal of the interface \( \partial V \). The periodic configuration implies that \( x_1 = \pm a/2 \) are equipotential lines and \( x_2 = \pm b/2 \) are lines of flow. That is, for the unit cell \( \Omega \), the following boundary conditions prevail:

\[
\varphi|_{x_1=\pm a/2} = \frac{1}{2} E a, \quad \varphi|_{x_1=\pm a/2} = -\frac{1}{2} E a, \quad \frac{\partial \varphi}{\partial x_2}\bigg|_{x_2=\pm b/2} = 0.
\]

Also, due to symmetry, we have the relations

\[
\varphi(x_1, x_2) = \varphi(x_1, -x_2),
\]

\[
\varphi(x_1, x_2) - \varphi(0, 0) = \left[ \varphi(-x_1, x_2) - \varphi(0, 0) \right].
\]

III. FORMULATION

A. Potential expansions

We first consider that the inclusion is purely cylindrically orthotropic, nongraded. The constitutive relation for cylindrically orthotropic cylinders is described by \( q = k_i H \). In matrix form, it is written as

\[
\begin{pmatrix} q_r \\ q_\theta \end{pmatrix} = \begin{pmatrix} k_r & 0 \\ 0 & k_\theta \end{pmatrix} \begin{pmatrix} -\partial \varphi/\partial r \\ -(1/r) \partial \varphi/\partial \theta \end{pmatrix},
\]

where the vector \( q \) is the (plane) heat flux vector expressed in a cylindrical coordinate system \((r, \theta)\), \( H = -\nabla \varphi \) is the thermal intensity, and \( k_r \) and \( k_\theta \) are the conductivities in the \( r \) and \( \theta \) directions, respectively. The potential field is governed by (1). For the cylindrically orthotropic inclusion, it is expanded as...
\[
\frac{\partial^2 \varphi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_i}{\partial r} + \frac{k_\mu k_i}{r^2} \frac{\partial^2 \varphi_i}{\partial \theta^2} = 0.
\]

For the isotropic matrix we have \(k_r = k_\theta\) and thus (7) simply reduces to the Laplace equation. In terms of polar coordinates, the general solution has the admissible form

\[
\varphi_i = C_0 + \sum_{n=1}^{\infty} C_{2n-1} r^{2n-1} \cos(2n-1) \theta, \quad k = \sqrt{k_\mu k_i},
\]

for the inclusion, and

\[
\varphi_m = A_0 + \sum_{n=1}^{\infty} \left[ A_{2n-1} r^{2n-1} + B_{2n-1} r^{-(2n-1)} \right] \cos(2n-1) \theta,
\]

for the matrix. The coefficients \(A_{2n-1}, B_{2n-1}\), and \(C_{2n-1}\) are some unknown constants to be determined. As we impose constant intensity along the \(x_1\) direction, all sine terms in (8) and (9) are thus not included. In addition, \(\varphi(r, \theta)\) needs to be antisymmetric with respect to the \(x_2\) axis, and thus only terms with an odd number are allowed. The constant coefficient \(A_0\) represents the reference potential, which can be set to zero for convenience. Thus we have \(A_0 = C_0 = 0\). To proceed, we impose the continuity conditions (3) at \(r = \rho\), which will render

\[
A_{2n-1} = T \rho^{-(2n-1)} B_{2n-1}, \quad C_{2n-1} = (T + 1) \rho^{-(2n-1)(1+k)} B_{2n-1},
\]

where

\[
T = \frac{k_m + \bar{k}}{k_m - \bar{k}}, \quad \bar{k} = \sqrt{k_\mu k_\theta}.
\]

Next we consider that the cylindrically orthotropic inclusion is exponentially graded along the radial direction,

\[
k_i(r) = \begin{pmatrix} k_r & 0 \\ 0 & k_\theta \end{pmatrix} \exp(\beta r),
\]

where \(\beta\) is a given real constant. Of course when \(\beta = 0\), the conductivity reduces to that of a purely cylindrically orthotropic one. The assumption of an exponentially varying property is common in the engineering literature in modeling functionally graded materials. Relevant works include those of Giannakopoulos and Suresh,\(^{11}\) Martin et al.,\(^{12}\) Kuo and Chen,\(^{3}\) and the references contained therein. Analogous to the derivation of (7), the potential field in the inclusion is governed by

\[
\frac{\partial^2 \varphi_i}{\partial r^2} + \left( \beta + \frac{1}{r} \right) \frac{\partial \varphi_i}{\partial r} + \frac{k_\mu k_i}{r^2} \frac{\partial^2 \varphi_i}{\partial \theta^2} = 0.
\]

Setting \(\varphi_i = R(r) \cos n \theta\) in (13), we are led to

\[
R'' + \left( \beta + \frac{1}{r} \right) R' - \frac{\mu^2}{r^2} R = 0, \quad \mu = nk.
\]

Equation (14) is a linear second-order differential equation, which has a regular singularity at \(r = 0\), an irregular singularity at \(r = \infty\), and no others. This allows us to transform the equation as a form known as Whittaker’s equation.\(^{24}\) To resolve (14), we introduce the weight function\(^{25}\)

\[
w(r) = \exp \left[ \int \left( \beta + \frac{1}{r} \right) \, d\zeta \right] = r \exp(\beta r),
\]

and rewrite the dependent variable \(R(r)\) as

\[
R(r) = w(r)^{-1/2} R_e(x), \quad \text{with} \ x = -\beta r.
\]

This transformation allows us to eliminate the first derivative term of (14) and, at the same time, transform the differential operator into a self-adjoint form. By some algebraic manipulations, (14) can be recast as

\[
R_e''(x) + \left( \frac{1}{4} + \frac{1}{2x} + \frac{1 - \mu^2}{x^2} \right) R_e(x) = 0,
\]

and the corresponding solutions are\(^{26}\)

\[
R_{e, n}(x) = c_n M_{1/2, n}(x) + d_n W_{1/2, n}(x),
\]

where the coefficients \(c_n\) and \(d_n\) are arbitrary constants. Here \(M_{\eta, \mu}(x)\) and \(W_{\eta, \mu}(x)\) are Whittaker functions,

\[
M_{\eta, \mu}(x) = e^{-x/2} x^{\mu+1/2} M \left( \mu - \eta + \frac{1}{2}, 2\mu + 1; x \right),
\]

\[
W_{\eta, \mu}(x) = e^{-x/2} x^{\mu+1/2} U \left( \mu - \eta + \frac{1}{2}, 2\mu + 1; x \right),
\]

and \(M(\cdot)\) and \(U(\cdot)\) represent the confluent hypergeometric equations.\(^{26}\) As the function \(W_{\eta, \mu}(x)\) becomes singular as \(x \to -0^+\),\(^{27}\) the admissible potential in the inclusion simply takes the form

\[
\varphi_i(r, \theta) = C_0 + \sum_{n=1}^{\infty} C_{2n-1} r^{2n-1} M((2n-1)k, 2(2n-1)k + 1; -\beta r) \cos(2n-1) \theta,
\]

where the constants \(C_{2n-1}\) are some unknown constants. Again, due to symmetry, only the terms with an odd number are allowed to preserve the antisymmetry around the origin. For the matrix, the temperature potential again takes the expression (9). Analogous to (10), the continuity conditions at the interface will give two constraints between the coefficients

\[
A_{2n-1} = T_{2n-1} \rho^{-(2n-1)} B_{2n-1},
\]

\[
C_{2n-1} = \frac{(T_{2n-1} + 1) \rho^{-(2n-1)(1+k)}}{M((2n-1)k, 2(2n-1)k + 1; -\beta \rho)} B_{2n-1},
\]

where, in contrast to (10), the coefficients \(T_n\) and \(M_n\) now depend on the number \(n\)

\[
T_{2n-1} = \frac{k_m + \bar{k} M_{2n-1}}{k_m - \bar{k} M_{2n-1}}.
\]
\[ M_n = e^{\beta \rho} \left[ 1 + \frac{\beta \rho}{2n+1} M(nk+1,2nk+2; -\beta \rho) \right]. \tag{22} \]

We mention that, in deriving (21), we have made use of the identity\(^{28}\)
\[ \frac{d}{dx} M(a,c;x) = aM(a+1,c+1;x). \tag{23} \]

**B. Rayleigh’s framework**

To proceed, the potential fields need to comply with the periodic conditions (4). For doing this, we impose the consistency condition of Rayleigh’s identity.\(^1\) Note that, for the present system which consists of cylindrically orthotropic and/or exponentially graded inclusions, the consistency condition remains the same as the systems with isotropic inclusions. This can be perceived from the fact that, owing to the periodic arrangements of the cylinders, the identity can be constructed by requiring that the potential in the matrix due to the source of the central cylinder be equal to the sum of the sources from the remaining lattice sites by constant shifts of the origin. Apparently, the properties of the inclusion do not affect the identity. Nicorovici et al.\(^{16}\) considered a composite medium consisting of a square array of coated cylinders and constructed the Rayleigh identity through a different approach. Specifically, they constructed the field potentials from two parts. One is due to the applied field at the remote boundary and the other is contributed by the induced charge which originated from the discontinuities of the normal component of electric displacement at the surface of each cylinder. They reached a similar conclusion that the identity for coated inclusions is exactly the same as that for noncoated inclusions. In fact, the approach of Nicorovici et al.\(^{16}\) can be extended to multicoated inclusions and even to generally graded inclusions along the radial direction. But the detailed formulation could be rather cumbersome. Specifically, the consistency condition takes the form\(^1\)
\[ \sum_{n=1}^{\infty} A_{2n-1} r_j^{2n-1} \cos(2n - 1) \theta \]
\[ = -E x_1 + \sum_{j \neq 0} \sum_{n=1}^{\infty} B_{2n-1} r_j^{(2n-1)} \cos(2n - 1) \theta_j, \tag{24} \]

where the index \( j \) runs over all cylinders except the one at the origin, \((r, \theta)\) is the polar coordinate measured from the center of the central cylinder \( O \), and \((r_j, \theta_j)\) is the polar coordinate of the point \((r, \theta)\) when measured in the local coordinate centered at the \( j \)th cylinder. Note that the identity is also valid for a hexagonal array of cylinders, if we change the allowed values of \( n \) in (9) and (8) according to the symmetries of the potential function. As in Nicorovici et al.\(^{16}\), one can use the addition theorem and rewrite the right-hand side of (24) as a function of \( r \) and \( \theta \), or by equating the \((2n-1)\)th partial derivative with respect to \( x_1 \) on both sides of (24) at the point \( O \). This will allow us to construct an infinite set of linear equations, valid for all positive integral values of \( n \),

\[ A_{2n-1} + \sum_{m=1}^{\infty} \left( \frac{2m + 2n - 3}{2n - 1} \right) \sum_{2m+2n-2} B_{2m-1} = -E \delta_{n,1}. \tag{25} \]

Here \( \binom{m}{n} \) is the binomial coefficient, \( \delta_{n,1} \) is the Kronecker delta symbol, and the quantities
\[ \sum_m = \sum_{j \neq 0} r_j^m \cos m \theta_j, \tag{26} \]

are the lattice sums. Again, \((r_j, \theta_j)\) is the polar coordinate of the central point \( O \) when measured in the local coordinate centered at the \( j \)th cylinder and the index \( j \) runs over all cylinders’ centers underlying the periodic array except the central one. The above infinite set of linear equations (25) together with (10) or (21) will suffice to determine the expansion coefficients \( A_{n}, B_{n} \), and \( C_{n} \) by truncation at some finite order. The result can be written in the matrix form\(^{16}\)
\[ (T + W)x = Mx = u. \tag{27} \]

where the matrix \( T \) is a diagonal matrix with the diagonal entries being the constant \( T \) of (11) for cylindrically orthotropic cylinders, and with the diagonal entries being \( T_{2n-1} \) of (22) for exponentially graded cylinders. The vector \( u \) is a column vector with only one nonzero component \( u_t = -\rho \) and \( x \) is a column vector whose components are \( x_{2m-1}, \ m = 1, 2, \ldots \). The matrix \( W \) and the unknown vector \( x \) has the elements
\[ W_{mn} = -\frac{(2m + 2n - 3)! \sum_{2m+2n-2} \rho^{2m+2n-2}}{(2m - 2)! (2n - 2)! \sqrt{2m - 1} \sqrt{2n - 1}}, \]
\[ x_{2m-1} = \sqrt{2m - 1} B_{2m-1} (E \rho^{2m-1}). \tag{28} \]

In later numerical illustrations, we shall demonstrate that the coefficients of the unknown vector \( x \) are in general nonzero. This reflects that the field intensity inside the inclusion is not uniform. This phenomenon is distinct from that predicted by the conventional micromechanical models, such as the dilute approximation, the Mori-Tanaka method, and the self-consistent method. These latter methods invoke the Eshelby’s solution\(^{20}\) and thus the intensity field inside the inclusion is spatially uniform. Furthermore, we note that the system of equations in (27) is formally the same as that for a system with isotropic cylinders. The difference appears only in the diagonal matrix \( T \). For isotropic inclusions with conductivity \( k_i \), the matrix \( T \) is simply a scalar multiple of an identity matrix with the scalar being \((k_n + k_i)/(k_n - k_i)). This seems to suggest that the cylindrically orthotropic inclusions can be exactly simulated by homogeneous isotropic ones with an equivalent isotropic conductivity

\[ k_i = \sqrt{k_r} k_{\rho}. \tag{29} \]

This is an interesting finding as the potential field in the inclusions is composed of various orders of positions and yet a single equivalent conductivity (29) would suffice for conditions containing various orders of positions. In Sec. IV, we shall elaborate this point further. Specifically, through a simple concept, we shall show that the geometric mean \( \sqrt{k_r k_{\rho}} \) is indeed capable of characterizing the behavior of cylindri-
ally orthotropic inclusions and that this replacement is applicable to a much wider scope.

As defined in (26), the lattice sum $\Sigma_m$ can be interpreted as the potential contributed by the whole inclusions except the central one, and hence depends on the absolute length of the unit cell. For convenience, we may introduce a normalized lattice sum $S_m$

$$S_m = \sum_{j \neq o} (r/ja)^m \cos m\theta,$$  

which assumes that the length of the cell parallel to the $x_1$ axis is equal to unity. The relation between $\Sigma_m$ and $S_m$ is then connected by $\Sigma_m = a^{-m}S_m$. Previous studies have reported that the sum $S_2$ is conditionally convergent and its value depends upon the shape of the exterior boundary of the array. For a square array $S_2 = \pi$. The lattice sums for $m > 2$ are absolutely convergent and can be obtained by direct summation. A list of nonzero lattice sums can be found in Ref. 14.

C. Effective conductivity

Our next task is to derive the effective conductivity from the potential field. The major distinction with previous studies is that the inclusions are cylindrically orthotropic and functionally graded. We first recall the basic definition of the effective conductivity $k^*$ given by

$$\langle q \rangle = k^* \langle H \rangle,$$  

where the angular brackets denote the volume averages over the unit cell $\Omega$,

$$\langle q \rangle = \frac{1}{\Omega} \int_{\Omega} q \, dx, \quad \langle H \rangle = \frac{1}{\Omega} \int_{\Omega} H \, dx.$$

To proceed, using the divergence theorem, Eq. (4), and the continuity condition of $\varphi$, it is readily seen that $\langle H \rangle = EI$. Next, to derive $\langle q \rangle$, we can write

$$D = \frac{T_1 T_3 + T_3 S_2 f \xi \pi^{-1} + 10T_1 S_6 f^3 \xi^3 \pi^{-3} + (10S_2 S_6 - 3S_4) f^4 \xi^4 \pi^{-4}}{T_3 + 10S_6 f^3 \xi^3 \pi^{-3}},$$

if the matrix $M$ is taken as a $2 \times 2$ matrix, and is obtained as

$$D = \frac{T_1 T_3 + T_3 S_2 f \xi \pi^{-1} + 10T_1 S_6 f^3 \xi^3 \pi^{-3} + (10S_2 S_6 - 3S_4) f^4 \xi^4 \pi^{-4} + 126T_1 T_3 S_{10} f^5 \xi^5 \pi^{-5} + (126S_2 S_{10} - 5S_4) f^6 \xi^6 \pi^{-6} + (126S_2 S_{10} - 735S_5 S_{10} - 735S_2 S_{28} - 378S_2 S_{26} + 210S_4 S_5 S_8 - 50S_4 f^{10} \rho \sigma \pi^{-9} / \Delta)}{T_3 + 10S_6 f^3 \xi^3 \pi^{-3}},$$

if the matrix $M$ is truncated as a $3 \times 3$ matrix. Although the expression for $D$ becomes lengthy when one considers higher orders, it exhibits no difficulty to evaluate the solution numerically.

$$\langle q \rangle = \frac{1}{\Omega} \int_{\Omega} q \, dx = \frac{1}{\Omega} \int_{\Omega} (x_i q_{ij}) \, dx = \frac{1}{ab} \int_{\partial \Omega} x_i q_{ni} \, ds.$$  

Note that in deriving (33), it is required that the flux vector be divergenceless and the normal component of the flux be continuous across $\partial \Omega$. No further conditions are imposed on the inclusion’s properties. Since $q_{ni} = 0$ along $BC$ and $AD$, we have

$$\langle q \rangle = \frac{k_m}{b} \int_{AB} \frac{\partial q_{mj}}{\partial x} \, ds.$$  

The effective conductivity $k_{11}$ is then found from the ratio $\langle q_{11} \rangle / E$. To proceed, we adopt Rayleigh’s approach which employed the Green’s second identity over the matrix region $\Omega$. This will give [Eq. (6) of Rayleigh]

$$-ab\langle q_{11} \rangle + abE + 2\pi B_1 = 0,$$

and hence

$$k_{11} = \frac{\langle q_{11} \rangle}{\langle H_{11} \rangle} = \frac{k_m}{1 + \frac{2f}{\rho^2 E / B_1}}, \quad f = \frac{\pi \rho^2}{ab}.$$  

To evaluate the quantity $B_{11}/E$, we need to truncate the matrix $M$ in (27) at some finite order and the effective conductivity $k_{11}$ of the composite is given by

$$k_{11} = 1 - 2f (M^{-1})_{11} = 1 - \frac{2f}{D}.$$  

Here $(M^{-1})_{11}$ represents the $(1, 1)$ entry of the matrix $M^{-1}$. If one truncates the matrix $M$ as a $1 \times 1$ matrix, the coefficient $D$ has the form

$$D = T_1 + S_2 f \xi \pi^{-1},$$

where $\xi$ is the aspect ratio of the rectangle $b/a$. For a square array, $\xi = 1$ and $k_{11}$ becomes

$$k_{11} = 1 - \frac{2f}{T_1 + f},$$  

which exactly agrees with the Maxwell-Garnett formula. The coefficient $D$ is obtained as

$$\Delta = T_1 T_3 + 10T_3 S_6 f^3 \xi^3 \pi^{-3} + 126T_1 T_3 S_{10} f^5 \xi^5 \pi^{-5} + (126S_2 S_{10} - 735S_5 S_{10} - 735S_2 S_{28} - 378S_2 S_{26} + 210S_4 S_5 S_8 - 50S_4 f^{10} \rho \sigma \pi^{-9} / \Delta),$$

if the matrix $M$ is truncated as a $3 \times 3$ matrix.
IV. NEUTRAL INCLUSION TO FIELDS OF ARBITRARY ORDERS

To prove that the equivalent isotropic conductivity (29) can indeed characterize the effect of the cylindrical orthotropy of the cylinders, we employ the concept of neutral inclusions. A neutral inclusion in conduction is defined as a foreign body which can be introduced in a host solid without disturbing the temperature field in it.\(^{31}\) The idea of neutrality dated back to half a century ago by Mansfield,\(^{32}\) who showed that a hole in an elastic medium can be reinforced as that for the uncut medium, and has received renewed attention in the last few years. For a detailed exposition of the context and related references, the reader is referred to the book of Milton.\(^2\) Perspectives and insights based on the concept have been reported in a number of applications, e.g., in Saint-Venant’s torsion.\(^{33}\) In most of these developments the temperature field in the surrounding medium assumes a linear function of positions. This, in fact, corresponds to the homogeneous boundary condition that is related to the determination of effective moduli of composites. Here we only request that the field outside be harmonic. A boundary condition containing various orders of positions is permitted. Homogeneous boundary fields are of course the simplest kind.

We now consider a cylindrically orthotropic cylinder which is to be introduced inside a matrix medium with isotropic conductivity \(k_c\). Suppose that the matrix fields are harmonic, we ask what the suitable conductivity of the matrix is so that after the insertion of the cylindrically orthotropic cylinder inside the matrix the field in the matrix will not be disturbed. To do the analysis, we first recall that the general harmonic field in the matrix can be written as

\[
\sum_{n=0}^{\infty} r^n (\tilde{P}_n \cos n\theta + \tilde{Q}_n \sin n\theta), \quad r < \rho^+,
\]

for some referenced origin \(O\), with \(\tilde{P}_n\) and \(\tilde{Q}_n\) being some given coefficients. Let us now introduce a cylindrically orthotropic cylinder, with conductivities given in (6), centered at \(O\). Now we wish to replace the circular region of the matrix \(r = \rho\) with a cylindrically orthotropic cylinder without disturbing the field and flux in the neighboring matrix \(\rho < r < \rho^+\). To do this, we first recalled that the admissible potential of the cylindrically orthotropic cylinder has the form (8). To fulfill these conditions, we need to impose the continuities of potential and normal component of flux across the interface \(r = \rho\). That is,

\[
\left. \sum_{n=0}^{\infty} r^n (P_n \cos n\theta + Q_n \sin n\theta) \right|_{r=\rho} = \left. \sum_{n=0}^{\infty} r^n (\tilde{P}_n \cos n\theta + \tilde{Q}_n \sin n\theta) \right|_{r=\rho},
\]

for some referenced origin \(O\), with \(\tilde{P}_n\) and \(\tilde{Q}_n\) being some given coefficients.

\[
k \sum_{n=0}^{\infty} nr^{n-1} (P_n \cos n\theta + Q_n \sin n\theta) \bigg|_{r=\rho} = k_c \sum_{n=0}^{\infty} nr^{n-1} (\tilde{P}_n \cos n\theta + \tilde{Q}_n \sin n\theta) \bigg|_{r=\rho}.
\]

These two equations are consistent if and only if the matrix’s conductivity fulfills the unique constraint

\[
k_c = k \sqrt{k_c k_\theta}.
\]

Note that (45) is independent of \(n\) and hence the consistency condition is a neutrality condition for fields of arbitrary orders. This proof explains our observation in Sec. III on why the periodic composites of cylindrically orthotropic cylinders can be effectively represented by periodic arrays of equivalent isotropic cylinders with conductivity \(k_c\). Note, however, that the equivalence (45) is indeed applicable to a much wider scope, not only restricted to the present context. For example, in a recent study on Saint Venant’s torsion of isotropic composite shafts, Ting et al.\(^{34}\) showed that cylindrical orthotropic cylinders can serve as building bricks of solvable microgeometry for a host shaft of shear rigidity \(\mu_c = \sqrt{\mu_x \mu_y}\). An earlier study on the effective antiplane shear modulus of...
composites reinforced with cylindrically orthotropic fibers also has the similar finding, but the latter proof was based on the assumption of homogeneous boundary conditions.

V. RESULTS AND DISCUSSIONS

Here we present numerical results for the potential fields and effective conductivities of the composite. To check the correctness of the analysis, we first assume that the cylinders are isotropic and compare our numerical calculations on the effective conductivity with the existing solutions. Next, we consider the system with cylindrically orthotropic cylinders. The first part is to check the convergence of the solutions. To do this, we check whether the numerical results fulfill the periodic boundary conditions (4). In the numerical calculation, we consider a system with $k_m=1$, $k_r=2$, and $k_g=8$. We plot in Fig. 2 the variation of the potential along the equipotential line $x_1=-a/2$ and the heat flux along the flow line $x_2=b/2$. In the calculation, the linear system (27) is truncated at $m=10$ and $m=40$. In comparison of the exact boundary condition, the error is within the range of $10^{-4}$ for the potential and within $10^{-3}$ for the heat flux. The maximum error occurs at the corner points, as anticipated. In Fig. 3, we demonstrate the potential contour for a system with cylindrically orthotropic cylinders. We find that if $k_r=k_g$, namely, the inclusions become isotropic, then potential fields inside the inclusions are nearly linear functions of $x_1$. It should be mentioned that higher-order terms are, in fact, nonvanishing, but its effect is rather minor compared with the linear term. The potential contours for the cases of $k_r<k_g$ and $k_r>k_g$ are rather distinct. One tends to deviate from the origin and the other moves towards the center of the cylinder. In Fig. 4, we illustrate the variation of $\beta$ versus the effective conductivity for various ranges of the volume concentration. When $\beta=0$, the inclusion becomes homogeneous. The shell of the cylinder is more conductive than that of the core for the cases of $\beta>0$, and conversely for $\beta<0$. In Fig. 5, we plot the potential contour for a system with isotropic, exponentially graded cylinders. Two different grading factors, $\beta=5$ and $\beta=-5$, are illustrated. The main difference of the contour pattern appears in the inclusions.

VI. CONCLUDING REMARKS

In summary, we have extended Rayleigh’s formulation to composites consisting of periodic arrays of cylindrically orthotropic and exponentially graded cylinders. The admissible potentials for the cylinder and the matrix are derived and calculated within sufficient accuracy. We discuss the applicability of the Rayleigh’s identity for the exponentially orthotropic cylinders.
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graded and curvilinearly anisotropic inclusions. We show how to estimate the effective conductivity from the potential field. In future studies, the present formulation can be generalized to incorporate a few different aspects. For example, one can allow that the cylinders are simply coated, multiply coated, or even continuously graded. The work of Ting et al.\(^\text{15}\) contains some relevant formulation for the subject. One can also consider that the interfaces between the cylinders and the matrix are imperfectly bonded. Another possible subject is to explore the transport behavior of periodic composites with spherically anisotropic and/or graded particles.

1L. Rayleigh, Philos. Mag. 34, 481 (1892).
28M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (Dover, New York, 1965), p. 507.