行政院國家科學委員會專題研究計畫 期中進度報告

非均質材料：聖維南扭轉與傳導(1/3)

計畫類別：個別型計畫
計畫編號：NSC91-2211-E-006-085-
執行期間：91年08月01日至92年07月31日
執行單位：國立成功大學土木工程學系（所）

計畫主持人：陳東陽

報告類型：精簡報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中華民國92年6月2日
行政院國家科學委員會補助專題研究計畫 □成果報告 □期

中進度報告

非均質材料：聖維南扭轉與傳導(1/3)

計畫類別：☑ 個別型計畫 □ 整合型計畫
計畫編號：NSC  91- 2211 - E - 006 - 085 -
執行期間： 91 年 8 月 1 日至 92 年 7 月 31 日

計畫主持人： 陳東陽
共同主持人：
計畫參與人員：

成果報告類型(依經費核定清單規定繳交)：☑ 精簡報告  □ 完整報告

本成果報告包括以下應繳交之附件：
☑ 赴國外出差或研習心得報告一份
☑ 赴大陸地區出差或研習心得報告一份
☑ 出席國際學術會議心得報告及發表之論文各一份
☑ 國際合作研究計畫國外研究報告書一份

處理方式：除產學合作研究計畫、提升產業技術及人才培育研究計畫、列管計畫及下列情形者外，得立即公開查詢
Abstract

This year we have completed two subjects. (I) Exact connections of warping fields and torsional rigidities of symmetric composite bars, and (II) Torsion of an isotropic shaft of arbitrary cross-section imbedded with multicoated or graded circular cylinders of cylindrically orthotropic materials.

Keywords: Saint-Venant torsion, torsional rigidity, symmetric sections.

1 Introduction

In (I), Saint-Venant's torsion of symmetric cylindrical bars consisting of two or four homogeneous phases is studied. A symmetric section is meant that the cross section of the cylindrical bar possesses reflectional symmetry with respect to one or more axes. Each constituent region may have different shear modulus. The idea of the analysis is to superimpose suitably reflected potentials to obtain the torsion solution of the same composite section but with different moduli. For two-phase sections, we show that, if the warping fields for a given symmetric section with phase shear moduli $\mu_1$ and $\mu_2$
are known a priori, then the warping fields for the same configuration but with a different set of constituent moduli $\mu'_1$ and $\mu'_2$ are readily found through simple linear superpositions. Further, suppose that the torsional rigidities $T(\mu, \mu')$ and $T(\mu'_1, \mu'_2)$ for any two sets of phase moduli can be measured by some experimental tests or evaluated through numerical procedures, then the torsional rigidity for any other combinations of constituent moduli $T(\mu', \mu'_2)$ can be exactly determined without any recourse to the field solutions of governing differential equations. Similar procedures can be applied to a 4-phase symmetric section. But the coefficients of superposition are only found for a few branches. Specifically, we find that depending on the conditions of $\mu$ and $\mu'$, admissible solutions can be divided into three categories. When the correspondence between the warping field is known to exist, a link between the torsional rigidities can be established as well.

In (II), finding a geometric configuration that is amenable to an exact determination or characterization of the torsional rigidity is a relatively new territory that has only recently begun to be explored. For example, a circular cross section with an assemblage of composite cylinders was only recently known to be an exactly solvable microgeometry. A host shaft with arbitrary cross section, equivalent to higher orders of boundary data, necessitates that the coated cylinder must be sufficiently multiply coated. The present analysis is to provide a theoretical framework showing how to design a neutral cylinder with any number of coatings or with graded shear rigidities in a cross-section under torsion. Specifically we consider that the constituents are cylindrically orthotropic with the shear moduli $\mu'_c$ and $\mu'_t$. The host shaft is isotropic with the shear modulus $\mu_o$. A simple and unified mathematical framework is first proposed for the analysis of a multicoated cylinder. It is proven that only a two by two matrix, resulting from a serial multiplication of matrices of the same order, will enter into the resulting expression. Next, the multicoated cylinder, which consists of piecewise constant shear rigidities, is generalized into a graded cylinder, with a continuous variation of the shear rigidity along the radial direction. We find that the warping field of a neutral graded cylinder, with varying radial and tangential shear rigidities, is governed by a second-order, homogeneous, ordinary differential equation. The method of Frobenius is adopted to obtain series solutions
for the warping functions. A remarkable result for the neutrality of the imbedded cylinders (multicoated or graded) is that when the harmonic mean of the shear moduli $\mu_\theta = \sqrt{\mu_1 \mu_2}$ is identical to the shear modulus $\mu_\theta$ of the host shaft, the neutrality condition is satisfied for any cross section of the host shaft. Finally, a condition is given for the torsional rigidity of the host shaft to remain the same with the inclusion of the imbedded cylinders.

2 Main results (I)

We consider a two-phase composite cross-section $\Omega$. The constituent regions are denoted by $\Omega_1$ and $\Omega_2 (\Omega = \Omega_1 \cup \Omega_2)$ which are reflectionally symmetric with respect to the interface $J$. The region $\Omega_1$ is occupied by material 1 with constant shear modulus $\mu_1$, and similarly $\Omega_2$ for material 2 with shear modulus $\mu_2$. Apart from the constraint that regions 1 and 2 must be geometrically symmetric with respect to $J$, no other restrictions on the geometric shapes of $\Omega_1$ (or $\Omega_2$) are imposed.

Suppose the cross section $\Omega$ is subjected to a Saint-Venant torsion, we let $w_i(z) = \varphi_i(z) + i \psi_i(z)$ denote the complex potential within $\Omega_i$. Let us now consider new complex potentials $w'_i$ in regions $\Omega_i, i = 1, 2$, which are certain linear combinations of the old potentials in the two regions via suitable reflection:

$$w'_1(z) = \varphi'_1(z) + i \psi'_1(z) = a_{11} w_1(z) + a_{12} \overline{w_2(z)}, \quad z \in \Omega_1,$$

$$w'_2(z) = \varphi'_2(z) + i \psi'_2(z) = a_{21} \overline{w_1(z)} + a_{22} w_2(z), \quad z \in \Omega_2$$

(2.1)

in which $a_{11}, a_{12}, a_{21}$ and $a_{22}$ are some coefficients. We prove that the coefficients can be exactly determined as

$$a_{11} = a_{22} = \frac{\mu'_1 \mu_2 - \mu'_1 \mu_1}{(\mu'_2 + \mu'_1)(\mu_2 - \mu_1)}$$

$$a_{12} = a_{21} = -\frac{\mu'_1 \mu_2 - \mu'_1 \mu_1}{(\mu'_2 + \mu'_1)(\mu_2 - \mu_1)}$$

(2.2)

This result is indeed surprisingly simple, which indicates that if the complex potentials for a given two-phase symmetric section with phase moduli $\mu_1$ and $\mu_2$ are known a priori, then the solutions for the same geometric configuration but with different moduli $\mu'_1$ and $\mu'_2$ are readily found by linear combinations of the potentials through (2.1) and (2.2). The connection (2.1) is valid pointwise inside the regions. In other words, the warping displacements of the new section can be exactly expressed
in terms of those of the old one.

We have shown that the warping fields for two-phase symmetric sections and for some four-phase symmetric sections can be related to the warping fields of the same cross-section, but with different phase moduli. Since $\Omega_1$ and $\Omega_2$ are symmetric with respect to the x axis, one can set

$$2\iint_{\Omega_1} \psi_1(z)dxdy = 2\iint_{\Omega_2} \psi_1(\bar{z})dxdy = A$$

$$2\iint_{\Omega_1} \psi_2(z)dxdy = 2\iint_{\Omega_2} \psi_2(\bar{z})dxdy = B$$

(2.3)

$$\iint_{\Omega_1} z\bar{z} dxdy = \iint_{\Omega_2} z\bar{z} dxdy = J$$

we have shown that

$$T(\mu_1, \mu_2) = \mu_1 A + \mu_2 B - (\mu_1 + \mu_2)J$$

$$T(\mu'_1, \mu'_2) = (\mu'_1 a_{11} - \mu'_2 a_{21}) A + (\mu'_2 a_{22} - \mu'_1 a_{12}) B - (\mu'_1 + \mu'_2)J$$

(2.4)

These formulae suggest that if the quantities $A, B$ and $J$ are known a priori, then the torsional rigidity of one composite section can be directly linked with that of the other. Suppose that one can measure or evaluate the torsional rigidities $T(\mu_1, \mu_2)$ and $T(\mu'_1, \mu'_2)$ through some experimental means or by numerical procedures, then the linear algebraic set of equations (2.4) will give the values of $A$ and $B$ without recourse to the solutions of partial differential equations. For potential applications, we can consider a third composite section with the same geometry but with a different combination of phase moduli $\mu''_1$ and $\mu''_2$. Analogous to (2.1), one can assume the complex potentials of the third section by linear superpositions from those of the first one

$$w'_1(z) = b_{11} w_1(z) + b_{12} w_2(\bar{z}), \quad z \in \Omega_1,$$

$$w''_2(z) = b_{21} w_1(\bar{z}) + b_{22} w_2(z), \quad z \in \Omega_2,$$

(2.5)

The coefficients $b_{ij}$ follow from (2.2) simply by changing $\mu'$ to $\mu''$. Similar to (2.4), its torsional rigidity has the form

$$T(\mu''_1, \mu''_2) = (\mu''_1 b_{11} - \mu''_2 b_{21}) A + (\mu''_2 b_{22} - \mu''_1 b_{12}) B - (\mu''_1 + \mu''_2)J$$

(2.6)

3 Main result (II)

We consider a circular multicoated cylinder $\Sigma$, with outer radius $a_1$, which is to be imbedded inside the host shaft at a selected location, say $\hat{x} = (\hat{x}, \hat{y})$. The multicoated
cylinder consists of a core with radius $a_n$ and $(n - 1)$ layers of coating. The $\alpha$th layer of the coating occupies the annulus $a_{\alpha+1} \leq r \leq a_\alpha$, $\alpha = 1, 2, \cdots, n - 1$. We assume that each constituent layer of the composite cylinder is cylindrically orthotropic. Under an applied torque at the ends of the cylindrical shaft, we ask what is the suitable shear rigidity of the host shaft $\mu_0$, or equivalently, what are the relationships between the phase rigidities and area fractions of the multicoated inclusion, so that after the insertion of the multicoated cylinder $\Sigma$ inside $\Omega$, the warping field and also shear traction on $\partial \Sigma$, and of course in $\Omega \setminus \Sigma$, remains unchanged as that of the unreinforced homogeneous shaft. We show that

$$\mu_0 = \frac{K^{(n,m)}_1 - K^{(n,m)}_2}{K^{(n,m)}_2 - K^{(n,m)}_1} \mu_G^{(l)}$$  \hspace{1cm} (3.1)

where

$$K^{(\alpha,m)}_m = \begin{bmatrix} c^{(\alpha)}_m & -c^{(\alpha)}_m \\ -c^{(\alpha)}_m h_a & c^{(\alpha)}_m h_a \end{bmatrix}, \quad c^{(\alpha)}_{\pm m} = (a_{\alpha+1} / a_\alpha)^{-m a} \pm (a_{\alpha+1} / a_\alpha)^{m a}$$  \hspace{1cm} (3.2)

and

$$K^{(\alpha,m)}_m = K^{(\alpha-1)}_m K^{(\alpha-2)}_m \cdots K^{(1)}_m$$  \hspace{1cm} (3.3)

When the multicoated cylinder is homogeneous, namely no coating ($n = 1$). It is obvious that the matrix $K^{(n,m)}$ does not involve. We find that the neutrality is satisfied if and only if

$$\mu_0 = \mu_G^{(l)} = \sqrt{\mu_r^{(l)} \mu_{\theta}^{(l)}}$$  \hspace{1cm} (3.4)

for any number of $m. This implies that for any given cross section $\Omega$ one can always introduce a cylindrically orthotropic cylinder, satisfying (3.4), inside a host shaft without disturbing the outer warping field in the host shaft.

Now when the number of constituent layers of the multicoated cylinder increases while each layer diminishes to infinitesimal thickness. In the limit the cylinder becomes graded with continuously varying radial and tangential shear rigidities, $\mu_r (r / a_1)$ and $\mu_\theta (r / a_1)$. Since a multicoated cylinder is capable of achieving the neutrality condition for any shape of $\Omega$, it is likely that one can also tailor the rigidity distribution of the graded cylinder to achieve the neutrality. In this section we shall explore the relation between the rigidity distribution of the graded cylinder and $\mu_0$ so that inserting a graded cylinder does not disturb the warping field in the host shaft under torsion. We find that the neutrality can be rewritten in the form

$$\hat{\gamma} \mu_0 (\hat{\gamma} \mu_0 A_m') - m^2 \mu_G^2 A_m = 0$$  \hspace{1cm} (3.5)

where
\[ \mu_G = \sqrt{\mu_r \mu_\theta}, \quad k = \sqrt{\mu_\theta / \mu_r}, \quad F = (\hat{r} \mu_r) / \mu_r \]  

(3.6)

with boundary conditions

\[ A_m \big|_{\hat{r}=1} = \tilde{A}_m, \quad \mu, A'_m \big|_{\hat{r}=1} = m \mu \tilde{A}_m \]  

(3.7)

In the case of multicoated cylinder, the neutrality condition is satisfied for any number of coating and for an arbitrary cross section of the host shaft if is holds. When \( \mu_G \) is a constant and

\[ \mu_0 = \mu_G = \sqrt{\mu_r (\hat{r}) \mu_\theta (\hat{r})} \]  

(3.8)

it can be shown that has the solution

\[ A_m = \tilde{A}_m e^{-m \mu \xi} \]  

(3.9)

that satisfies the condition (3.7) and the condition that \( A_m \) is bounded at \( \hat{r} = 0 \). Thus a neutrality condition is satisfied when (3.8) holds. Equation (3.8) tells us that only one of \( \mu_r (\hat{r}) \) and \( \mu_\theta (\hat{r}) \) can be prescribed arbitrarily.

4 Concluding Remarks

Neutral cylinders can serve as useful building bricks of an exactly solvable microgeometry for a host shaft under torsion. Depending on the cross-sectional shape of the host shaft, a neutral cylinder needs to be suitably multicoated. In this work we presented a unified, exact approach to characterize the criteria to achieve neutrality. Only a two by two matrix, resulting from a serial multiplication of matrices of the same order, is involved in the final expression. A graded cylinder with continuously varying shear rigidities is another kind of cylinders that are capable to accomplish the task. We show that the warping field of a neutral graded cylinder is governed by a linear, second-order, homogeneous, ordinary differential equation. The method of Frobenius is used to derive the solution field in series form. Remarkably we found that a multicoated cylinder can be a neutral cylinder in a host shaft of arbitrary cross section when the single condition is satisfied. For graded cylinders we prove analytically that a similar conclusion applies. Physically, these findings imply that the torsional rigidity of any cross-section filled with an assemblage of cylindrically orthotropic cylinders can be exactly analyzed.

References


<table>
<thead>
<tr>
<th>發明人/創作人</th>
</tr>
</thead>
<tbody>
<tr>
<td>技術說明</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>中文：</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>(100-500字)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>英文：</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>可利用之產業及可開發之產品</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>技術特點</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>推廣及運用的價值</td>
</tr>
<tr>
<td>可發表學術論文</td>
</tr>
</tbody>
</table>

※ 1. 每項研發成果請填寫一式二份，一份隨成果報告送繳本會，一份送貴單位研發成果推廣單位（如技術移轉中心）。
※ 2. 本項研發成果若尚未申請專利，請勿揭露可申請專利之主要内容。
※ 3. 本表若不敷使用，請自行影印使用。