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非線性控制理論與模擬

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非線性系統的平衡化處理Hamilton-Jacobi PDE 在工程上的應用

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Applications of Hamilton-Jacobi PDE

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ABSTRACT

In this paper we propose a pure $H_\infty$ guidance law for homing missiles with maneuvering targets. The nonlinear $H_\infty$ suboptimal control design governed by the Hamilton-Jacobi PDE inequality associated with this guidance problem is solved analytically. Based on the existing nonlinear $H_\infty$ techniques, we synthesize an $H_\infty$ guidance law which can pursue a class of maneuvering targets with acceptable energy consumption and miss distance.

I. INTRODUCTION

A system in the presence of unknown parameters or disturbances often has poor performance or even becomes unstable when the controllers are optimized with respect to a nominal model of the system or with respect to a fixed disturbance. Hence, robustness of performance with respect to model variations and/or disturbance variations must be considered in the control design. In recent years, the "robustness" idea has been exploited in many aerospace systems, such as missile autopilot design [1,2], aeroscanned orbital transfer [3], and attitude control of rigid spacecraft [4]. The robust guidance law considered in [5] took into account the uncertainty of the time constant $\tau$ within the guidance loop.

As one of the powerful robust control techniques, the $H_\infty$ control theory has been widely applied to various engineering systems with modeling uncertainties or exogenous disturbances. In aerospace applications, for example, a linear $H_\infty$ controller based on the linearization of a space station model was studied in [6], and a nonlinear $H_\infty$ control problem was solved for satellite attitude control in [4,7]. As a novel application, in this paper, we will apply the nonlinear $H_\infty$ control theory to synthesize a missile guidance law which is robust to the variation of target’s accelerations.

The nonlinear $H_\infty$ control problem studied in [8,9,10] involves the solution of the Hamilton-Jacobi (H-J) partial differential inequality which, in general, is difficult to solve. In [4,7], algebraic and geometric tools were used to find a particular solution of the H-J inequality for the satellite attitude control problem while in [11], a numerical algorithm was developed to find the Taylor series solution of the H-J inequality. As to the homing guidance problem considered here, fortunately, we find that the solution to the associated Hamilton-Jacobi inequality can be solved analytically.

In the analysis of proportional navigation (PN), it is often assumed that the target is nonmaneuvering or is maneuvering with known accelerations. In this paper, a significant point is that we remove the assumption that the target’s trajectories or accelerations are known. For various target’s maneuvers, the $H_\infty$ guidance law is found to maintain its interception performance with acceptable miss distance and energy consumption. Nonlinear relative motion is considered here, and the target’s accelerations are regarded as unpredictable disturbances. The missile guidance problem is then reformulated as a nonlinear disturbance attenuation control problem, where the purpose of the control design is to attenuate the influence of the target’s maneuvers on the performance of the guidance law, such as, the miss distance, energy consumption, etc. The conventional PN schemes, such as pure proportional navigation (PPN) [12], true proportional navigation (TPN) [13], and ideal proportional navigation (IPN) [14], have been compared with the proposed guidance law. It is observed from the simulation results that the conventional PN schemes do perform very well in the absence of target maneuvers; however, when target maneuvers are present, their performance worsens dramatically. The $H_\infty$...
guidance law may not perform better than the PN schemes for nonmaneuvering targets, but it demonstrates strong performance robustness under variation of target maneuvers.

This paper is organized as follows. In the next section, we will briefly survey some results of the state feedback nonlinear $H_{\infty}$ control theory. In section 3, the missile guidance problem will be reformulated as a nonlinear disturbance attenuation control problem. The associated H-J inequality will then be solved analytically in section 4 to obtain the nonlinear $H_{\infty}$ guidance law. Comparison with PN schemes and the robustness of the $H_{\infty}$ guidance law against target maneuvers will be illustrated numerically in section 6.

II. NONLINEAR $H_{\infty}$ CONTROL THEORY

In this section, we will introduce the standard results of the nonlinear $H_{\infty}$ control theory for later use. Consider a nonlinear state space system:

\begin{align}
\dot{x} &= f(x) + g(x)w, \quad w \in \mathbb{R}^n, \quad f(0) = 0 \\
\dot{z} &= h(x), \quad z \in \mathbb{R}^n, \quad h(0) = 0,
\end{align}

where $x$ is the state vector, and $w$ and $z$ are the exogenous disturbances to be rejected and the penalized output signal, respectively. We assume that $f(x)$, $g(x)$, and $h(x)$ are $C^r$ functions, and that $x = 0$ is the equilibrium point of the system, i.e., $f(0) = h(0) = 0$.

If there exists a scalar $C^1$ function $U: \mathbb{R}^n \to \mathbb{R}$ with $U(0) = 0$ such that

\[ \frac{1}{2}\gamma^2 \left[ \frac{\partial U}{\partial x} \right] g(x) \dot{g}(x) \left[ \frac{\partial U}{\partial x} \right]^T + \left[ \frac{\partial U}{\partial x} \right] f(x) + \frac{1}{2} \dot{h}^T(x) h(x) \leq 0, \]

then the system is said to have $L_2$ gain $\leq \gamma$, i.e.,

\[ \gamma \leq \int_0^\infty \gamma^2 dt \leq \int_0^\infty w^T w dt, \]

where $\frac{\partial U}{\partial x} = \left[ \frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \ldots, \frac{\partial U}{\partial x_n} \right]$. The details of the proof can be found, for example, in [9].

When control $u$ is applied to the system, we get the following controlled system:

\begin{align}
\dot{x} &= f(x) + g(x)w + g_z(x)u \\
\dot{z} &= h_1(x) + k_{12}(x)u,
\end{align}

shown later, these assumptions are satisfied automatically for the missile guidance problem. The nonlinear $H_{\infty}$ control problem is to find the control $u$ such that the $L_2$-gain of the closed-loop system is less than or equal to $\gamma$. By replacing $f(x)$, $g(x)$, and $h(x)$ in Eq. (2) with $f(x) + g_z(x)u$, $g_z(x)$, and $h_1(x) + k_{12}(x)u$, respectively, the condition that the $L_2$-gain of the closed-loop system is equal to or lower than $\gamma$ becomes

\[ \frac{1}{2} \left[ u^T + \frac{\partial U}{\partial x} g_z \right] \left[ u^T + \frac{\partial U}{\partial x} g_z \right]^T + \frac{1}{2} \left[ \frac{\partial U}{\partial x} \right] \left[ \frac{1}{\gamma^2} g_z g_z^T - \gamma^2 \left[ \frac{\partial U}{\partial x} \right]^T \right] \left[ \frac{\partial U}{\partial x} \right]^T + \left[ \frac{\partial U}{\partial x} \right] f + \frac{1}{2} \dot{h}_1^T h_1 \leq 0. \]  

The control $u$ minimizing the left-hand-side of the above inequality can be easily found as

\[ u(x) = -g_z^T(x) \left[ \frac{\partial U}{\partial x} \right]^T. \]

Substituting this control into Eq. (5), we obtain the desired HJPI as

\[ \left[ \frac{\partial U}{\partial x} \right] f + \frac{1}{2} \left[ \frac{\partial U}{\partial x} \right] \left[ \frac{1}{\gamma^2} g_z g_z^T - \gamma^2 \left[ \frac{\partial U}{\partial x} \right]^T \right] \left[ \frac{\partial U}{\partial x} \right]^T + \frac{1}{2} \dot{h}_1^T h_1 \leq 0. \]

Hence, solving the nonlinear $H_{\infty}$ control problem is equivalent to finding a positive function $U(x)$ satisfying HJPI. The corresponding HJPI for the missile guidance problem will be derived in the next section.

III. FORMULATION OF $H_{\infty}$ GUIDANCE PROBLEMS

For a plane intercept, relative motion between the missile and target is described by the polar coordinates system $(r, \theta)$ with the origin fixed at the missile initial location (refer to Fig. 1). For the purpose of guidance law design, the missile and target are assumed to be point masses, and only kinematics are considered. The governing equations of the relative motion can be derived as

\[ r - r_0 = w_r - u_r, \]

\[ r \dot{\theta} + 2 \dot{r} \theta = w_\theta - u_\theta, \]

where $r$ is the relative distance between the missile and the target; $\theta$ is the angle of the line-of-sight (LOS) with respect to an inertial reference line; $w_r$ and $w_\theta$ are the target's acceleration components, which are assumed to
be an unpredictable disturbance; \( u_r \) and \( u_{\theta} \) are the missile's acceleration components which are to be found. By introducing the new state variables \( (r, V_r, V_\theta) \), where \( V_r = r \) is the radial relative velocity and \( V_\theta = r \dot{\theta} \) is the tangential relative velocity, we can transform Eqs. (8) into the standard state space form as in Eq. (4a):

\[
\begin{bmatrix}
  \dot{r} \\
  \dot{V}_r \\
  \dot{V}_\theta
\end{bmatrix} =
\begin{bmatrix}
  V_r \\
  \frac{V_r^2}{r} + \frac{V_\theta}{r} \\
  -V_r \frac{V_\theta}{r}
\end{bmatrix} +
\begin{bmatrix}
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  0 & -1 & 0
\end{bmatrix} w +
\begin{bmatrix}
  0 \\
  -1 \\
  0
\end{bmatrix} u,
\]

(9)

where \( w = [w_r, w_\theta]^T \) and \( u = [u_r, u_\theta]^T \) are the acceleration vectors of the target and missile, respectively.

Next, we need to specify the output signal \( z \) to be controlled. A good guidance law must guarantee a decreasing relative distance and, at the same time, keep the LOS angular rate as small as possible. To reflect this fact, we choose \( z \) as

\[
z = \begin{bmatrix}
  \rho h(r, V_\theta) \\
  u
\end{bmatrix} = \begin{bmatrix}
  \rho h(r, V_\theta) \\
  0
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} u,
\]

(10)

where

\[
h(r, V_\theta) = \frac{V_\theta^2}{r} = r \dot{\theta}_2
\]

(11)

is a measure of guidance performance, and \( \rho \) is a weighting factor concerning the trade-off between performance and acceleration command. From the definition of \( h \), it can be seen that when \( h \) can be kept small, this implies that the missile is close to the target (small \( r \)), and/or that the LOS angular rate \( \dot{\theta} \) is small. By choosing a proper weighting factor \( \rho \), it is possible to obtain an acceptably small \( h \) without consuming a lot of acceleration \( u \).

The problem of \( H_{\infty} \) guidance law design now can be stated as follows: Find the missile acceleration command \( u = [u_r, u_\theta]^T \) such that the \( L_2 \)-gain of the system described by Eq. (9) and Eq. (10) is lower than or equal to \( \gamma \), i.e.,

\[
\begin{align*}
\int_0^\infty z^T z dt &= \int_0^\infty (\rho^2 h^2 + u^T u) dt \\
\int_0^\infty w^T w dt &= \int_0^\infty (w_r^2 + w_\theta^2) dt \\
&\leq \gamma^2, \forall w \in L_2.
\end{align*}
\]

(12)

In the above equation, \( \int_0^\infty w^T w dt \) is the input energy of the system, i.e., the \( L_2 \)-norm of the disturbance (target's acceleration) while \( \int_0^\infty z^T z dt = \int_0^\infty (\rho^2 h^2 + u^T u) dt \) denotes the output energy of the system, i.e., the \( L_2 \)-norm of the desired performance. We wish the output energy to be as small as possible under the action of the disturbance input \( w \). The system gain \( \int_0^\infty z^T z dt \int_0^\infty w^T w dt \) can, thus, be regarded as the disturbance attenuation level, and the \( H_{\infty} \) guidance law can guarantee that the disturbance attenuation level will be lower than or equal to a specified value \( \gamma \) for the target's acceleration \( w = [w_r, w_\theta]^T \in L_2 \). It is due to the above inherent property of the guaranteed disturbance attenuation level that the \( H_{\infty} \) guidance law can exhibit performance robustness against variation of target maneuvers.

**IV. MAIN RESULT**

Based on the above derivation, the nonlinear \( H_{\infty} \) guidance law is given by Eq. (9) as

\[
u(r, V_r, V_\theta) = -(U_d g)^T,
\]

(13)

where \( U \geq 0 \) is to be determined. Comparing Eq. (10), we can rewrite the Hamilton-Jacobi PDE as

\[
H = [U, U_{V_r}, U_{V_\theta}] 
\begin{bmatrix}
  V_r \\
  V_\theta^2 / r \\
  -V_r V_\theta / r
\end{bmatrix}
\]

\[
+ 1 / 2 (1 / \gamma^2 - 1) [U, U_{V_r}, U_{V_\theta}] 
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix} [U, U_{V_r}, U_{V_\theta}]
\]

\[
+ 1 / 2 \rho^2 h^2 (r, V_\theta) \leq 0.
\]

(14)

Expanding the above inequality leads to
\[
V_r \left( \frac{\partial U}{\partial r} \right) + V_{\theta}^2 \left( \frac{\partial U}{\partial \theta} \right) - \frac{V_r V_\theta}{r} \left( \frac{\partial U}{\partial \theta} \right) + \frac{1}{2} \left( \frac{1}{\gamma^2} - 1 \right) \left[ \left( \frac{\partial U}{\partial r} \right)^2 + \left( \frac{\partial U}{\partial \theta} \right)^2 \right] + \frac{\rho^2 V_\theta^4}{2 r^2} \leq 0 .
\]
(15)

It is difficult to find the solution \( U(r, V_r, V_\theta) \) of the above PDI inequality, in general. After examining the existing guidance laws, we find that no guidance law can satisfy the inequality (15). But it is interesting to find that the ideal proportional navigation (IPN) is close to the solution of the H-J inequality. The missile acceleration for IPN can be expressed as follows:

\[
\begin{bmatrix}
\dot{u}_r \\
\dot{u}_\theta
\end{bmatrix} = \begin{bmatrix}
N r \theta \gamma \theta \\
-N r \theta \gamma
\end{bmatrix} = \begin{bmatrix}
N V_\theta^2 / r \\
-N V_\theta / r
\end{bmatrix} .
\]
(16)

Unfortunately, there exists no solution \( U \) of the H-J inequality corresponding to IPN such that Eq. (6) is satisfied, i.e.,

\[
\begin{bmatrix}
\partial U / \partial V_r \\
\partial U / \partial V_\theta
\end{bmatrix} = \begin{bmatrix}
N V_\theta^2 / r \\
-N V_\theta / r
\end{bmatrix} .
\]
(17)

Although the \( U \) in Eq. (17) does not exist, Eq. (17) motivates the selection of the following function as the candidate solution of the H-J inequality (15):

\[
U(r, V_r, V_\theta) = -\frac{N V_r V_\theta^2}{r} , \quad N > 0 .
\]
(18)

where \( N \) is a constant to be determined. It is also noted that \( U(r, V_r, V_\theta) > 0 \) since \( r \geq 0 \) and \( V_r \leq 0 \).

Now, we want to determine the navigation constant \( N \) such that inequality (15) is satisfied. First, we substitute the partial derivative of \( U \) into inequality (15) and simplify the equation to obtain

\[
H = \left[ 3 N + 2 N^2 \left( \frac{1}{\gamma^2} - 1 \right) \right] \frac{V_r V_\theta^2}{r^2} + \left[ N + N^2 \left( \frac{1}{\gamma^2} - 1 \right) + \frac{\rho^2}{2} \right] \frac{V_\theta^4}{r^2} .
\]
(19)

To satisfy \( H \leq 0 \), we only need to impose the following conditions:

\[
3 N + 2 N^2 \left( \frac{1}{\gamma^2} - 1 \right) \leq 0
\]
(20a)

\[
- N + \frac{N^2}{2} \left( \frac{1}{\gamma^2} - 1 \right) + \frac{\rho^2}{2} \leq 0 .
\]
(20b)

Solving simultaneously the set of inequalities (20), we have the range of the navigation constant \( N \) as

\[
N \geq \frac{-3/2}{1 - 1/\gamma^2} , \quad \rho \leq \frac{1}{2} \sqrt{\frac{21}{1 - 1/\gamma^2}} \quad \text{(21a)}
\]

\[
N \geq \frac{-1 + \sqrt{1 + \rho^2 (1 - 1/\gamma^2)}}{1 - 1/\gamma^2} , \quad \rho > \frac{1}{2} \sqrt{\frac{21}{1 - 1/\gamma^2}} . \quad \text{(21b)}
\]

Recalling the definition of the output signal \( z \) for the system, we know that the magnitude of \( \rho \) reflects the relative importance between the guidance performance and the interceptive energy. Now, consider two limiting cases: (i) \( \rho \to 0 \), where only interceptive energy is concerned, and (ii) \( \rho \to \infty \), where only guidance performance is concerned:

\[
N \geq \frac{3/2}{1 - 1/\gamma^2} , \quad \rho \to 0 \quad \text{(22a)}
\]

\[
N \geq \frac{\rho}{\sqrt{1 - 1/\gamma^2}} , \quad \rho \to \infty . \quad \text{(22b)}
\]

From Eq. (3), we know that the larger \( \gamma \) is, the more likely it is that the input disturbance \( w \) will affect the performance of the missile. When \( \gamma \) approaches infinity, the \( H_\infty \) controller is reduced to the \( H_2 \) (LQG) controller [15]. Letting \( \gamma \to \infty \), we have from Eq. (21):

\[
N \geq \frac{3}{2} , \quad \rho \leq \sqrt{2 \Gamma} / 2 \quad \text{(23a)}
\]

\[
N \geq -1 + \sqrt{1 + \rho^2} , \quad \rho > \sqrt{2 \Gamma} / 2 . \quad \text{(23b)}
\]

As \( \gamma \) approaches one, the tracking ability of the missile increases, but at the same time, the navigation constant \( N \) approaches infinity, indicating that the control energy is very large. But when \( \gamma \) is equal to one, \( N \) does not exist. In other words, \( \gamma \) changes according to \( \rho \) and \( N \), and at the critical point \( \gamma = 1 \), we can't get an exact \( \gamma \) by regulating \( \rho \) and \( N \). Finally, substituting \( U(r, V_r, V_\theta) \) into Eq. (13), we can obtain the \( H_\infty \) guidance law as

\[
\begin{bmatrix}
\dot{u}_r \\
\dot{u}_\theta
\end{bmatrix} = \begin{bmatrix}
\partial U / \partial V_r \\
\partial U / \partial V_\theta
\end{bmatrix} = \begin{bmatrix}
-N V_\theta^2 / r \\
-N V_\theta / r
\end{bmatrix} .
\]
(24)

Applying the guidance law in Eq. (24) to Eq. (9), the governing equations of the relative motion under \( H_\infty \) guidance strategy become

\[
r = V_r , \quad \rho(0) = r_0 
\]
(25a)

\[
V_r = (N + 1) \frac{V_\theta^2}{r} + w_r , \quad V_\theta(0) = V_{\theta 0}
\]
(25b)
\[ V_\theta = (2N - 1) \frac{V_r V_\theta}{r} + w_\theta, \quad V_\theta(0) = V_{\theta_0}. \] (25c)

V. SPECIAL CASE

Now a special case of a target maneuver will be considered to illustrate the system performance in detail. Let \( \omega = 0 \) and \( \omega_\theta = 0 \), i.e., the target flies along a straight line. Thus, Eq. (25) in this case is reduced to

\[ r = V_r, \] (26a)

\[ V_r = (N + 1) \frac{V_\theta^2}{r}, \] (26b)

\[ V_\theta = (2N - 1) \frac{V_r V_\theta}{r}. \] (26c)

From the above equations, if we define the angular counterclockwise direction as positive and \( N > 1/2 \), we have \( V_r > 0 \) and \( V_\theta < 0 \); hence, \( V_r \) is monotonically increasing from \( V_{\theta_0} < 0 \) to \( V_{\theta_i} = 0 \), and \( V_\theta \) is monotonically decreasing from \( V_{\theta_0} > 0 \) to \( V_{\theta_i} = 0 \).

For this special case, the missile trajectory can be found analytically. First, dividing Eq. (26c) by Eq. (26b), we get

\[ \frac{dV_\theta}{dV_r} = \frac{N + 1}{2N - 1} \frac{V_r}{V_\theta}. \] (27)

Integrating both sides of Eq. (27), we have

\[ V_r^2 - \frac{N + 1}{2N - 1} V_\theta^2 = V_{\theta_0}^2 - \frac{N + 1}{2N - 1} V_{\theta_i}^2 = \text{constant}. \] (28)

On the other hand, from Eq. (26a), we have

\[ \frac{dV_\theta}{V_\theta} = (2N - 1) \frac{dr}{r}. \] (29)

After integration, we have

\[ V_\theta = V_{\theta_0} \left( \frac{r}{r_0} \right)^{2N - 1}. \] (30)

Substituting Eq. (30) into Eq. (28) and rearranging the result, we get

\[ V_r^2 - V_{\theta_0}^2 = \frac{N + 1}{2N - 1} V_{\theta_0}^2 \left( \frac{r}{r_0} \right)^{4N - 2} - 1. \] (31)

Setting \( r \) in Eq. (31) to 0, we have

\[ V_{r_i}^2 = V_{\theta_0}^2 - \frac{N + 1}{2N - 1} V_{\theta_0}^2 \geq 0. \] (32)

Examining Eq. (32), we see that if the zero miss distance exists, then the initial condition \( (V_{r_0}, V_{\theta_0}) \) needs to satisfy the following constraint:

\[ \left| \frac{V_{r_0}}{V_{\theta_0}} \right| \geq \frac{\sqrt{N + 1}}{2N - 1}. \] (33)

We call this inequality (33) the capture area. It determines the geometric relation when a missile directly hits a target. Having obtained the relations of \( V_r \) and \( V_\theta \), we are ready to evaluate the final time of interception for this \( H_\omega \) guidance law. The closing speed \( V_r \) is calculated first. From Eq. (31), we have

\[ r = -\sqrt{\frac{N + 1}{2N - 1} V_{\theta_0} \left( \frac{r}{r_0} \right)^{4N - 2} + V_{r_i}^2}, \] (34)

where \( V_{r_i} \) is given in Eq. (32). Integrating Eq. (34), we obtain

\[ t = \int_0^{r_0} \frac{dr}{\sqrt{N + 1} V_{\theta_0} \left( \frac{r}{r_0} \right)^{4N - 2} + V_{r_i}^2}}. \] (35)

The interception time \( t \) can be obtained by setting \( r = 0 \) in the lower limit of the above integration.

On the other hand, if condition (33) is not satisfied, the missile can't intercept the target with zero miss distance. In this case, we can set \( V_r = \dot{r} = 0 \) in Eq. (31) to obtain the miss distance as follows:

\[ r_{\min} = r_0 \left[ 1 - \frac{2N - 1}{N + 1} \left( \frac{V_{\theta_0}}{V_{r_0}} \right)^{2} \right]^{(4N - 2)}. \] (36)

From the above equation, we know that the larger the navigation constant \( N \) is, the smaller \( r_{\min} \) will be. When the initial condition is not located within the capture area, \( r_{\min} \) must be smaller than the explosive radius.

From Fig. 9, we can understand the meaning of the explosive area clearly. The trajectory 1 is not located in the capture area, so \( r_{\min} > 0 \), and the missile needs to be ignited when \( r = r_{\min} \). At the time, the target is destroyed by the strong shock wave of the exploding missile.

VI. NUMERICAL RESULTS

To illustrate the robust property of the \( H_\omega \) robust guidance law against the variation of target maneuvers, we will analyze the effects of different types of target maneuvers on the interception performance of the \( H_\omega \) guidance law. Four types of target maneuvers are considered here, namely, (i) a smart target [16] with acceleration components \( \omega = 0; \omega_\theta = \omega_\phi(r - \theta) \); (ii) a step target...
with acceleration components $w_x = 0$; $w_y = N_T$; (iii) a ramp target with acceleration components $w_x = 0$; $w_y = N_T$; and (iv) a sinusoidal target [17] with acceleration components $w_x = 0$; $w_y = N_T \sin w_t$, where $N_T$ is the target’s navigation gain, which is an uncertainty varying between 1 and 10. Four guidance laws will be compared, i.e., the $H_m$ guidance law (HGL), TPN, PPN, and IPN. Each guidance law will be tested using all four types of target maneuvers. The discussion of the simulation contains three parts: (A) the performance of the four guidance laws with respect to a nonmaneuvering target, (B) a comparison of the disturbance attenuation capability of the four guidance laws in the presence of the four types of maneuvering targets with uncertain navigation gains, and (C) the influence of the weighting factor $\rho$ on the performance of HGL.

(A) Nonmaneuvering targets: Fig. 2a and Fig. 2b depict the missile trajectories for nonmaneuvering targets. Figure 2a corresponds to the zero miss-distance case, where it can be seen that HGL uses the most time to intercept the target. Figure 2b corresponds to the nonzero miss-distance case, where the initial conditions do not lie within the capture area characterized by Eq. (36). HGL is found to produce the largest miss distance among the four guidance laws. It can be concluded that conventional PN schemes provide excellent interceptive strategy for nonmaneuvering targets while HGL exhibits awkward performance in this case. This result is not surprising since HGL is designed under the assumption that the target’s motion is unknown or uncertain. If the target’s motion is completely known, the performance of the robust guidance law may become too conservative, and the best guidance law can be obtained via the optimization process with respect to the given target’s trajectory.

(B) Comparison of disturbance attenuation capability: The disturbance attenuation level $\gamma$ is defined as

$$\gamma = \frac{\|z\|_2/\|w\|_2}{(\int_0^{\infty} z^2 dt)^{1/2}/(\int_0^{\infty} \dot{w}^2 dt)^{1/2}},$$

which is a measure of how the performance output $z$ is affected by the disturbance input $w$ (the target’s maneuvers). From the definition of $\gamma$ in Eq. (14), a small $\|z\|_2/\|w\|_2$ implies good interceptive performance and low energy consumption. Hence, when $\|z\|_2/\|w\|_2$ is small, this indicates that good performance can be preserved in the presence of large target maneuvers (i.e., large $\|w\|_2$). The disturbance attenuation levels for HGL, IPN, TPN, and PPN are plotted vs the target’s navigation gain in Fig. 3 to Fig. 6, respectively. Figure 3 shows that the disturbance attenuation level for HGL can be kept below 10 for each type of target maneuver with navigation gain $N_T$ varying between 1 and 10. Figure 4 shows the disturbance attenuation level for IPN, where it is observed that the disturbance attenuation capability for IPN is very poor for each type of target, especially for step targets and ramp targets. Even a target with a small navigation gain can cause the performance of a missile guided by IPN to
Table 1. Comparisons of the performance of the four guidance laws \( (N_T = 1) \).

<table>
<thead>
<tr>
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<th>Smart target</th>
<th>Step target</th>
<th>Ramp target</th>
<th>Sine Wave target</th>
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<tbody>
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<td>HGL</td>
<td>( r = 0.4186 )</td>
<td>( r = 0.3175 )</td>
<td>( r = 0.3228 )</td>
<td>( r = 0.3112 )</td>
</tr>
<tr>
<td></td>
<td>( u = 0.4576 )</td>
<td>( u = 0.4895 )</td>
<td>( u = 0.4995 )</td>
<td>( u = 0.4884 )</td>
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<tr>
<td>IPN</td>
<td>( r = 0.4576 )</td>
<td>( r = 0.3321 )</td>
<td>( r = 0.3381 )</td>
<td>( r = 0.3521 )</td>
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<tr>
<td></td>
<td>( u = 63.021 )</td>
<td>( u = 36.333 )</td>
<td>( u = 40.000 )</td>
<td>( u = 81.532 )</td>
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<tr>
<td>TPN</td>
<td>( r = 0.4238 )</td>
<td>( r = 0.3048 )</td>
<td>( r = 0.3568 )</td>
<td>( r = 0.3391 )</td>
</tr>
<tr>
<td></td>
<td>( u = 2.6228 )</td>
<td>( u = 18.0958 )</td>
<td>( u = 34.182 )</td>
<td>( u = 3.2364 )</td>
</tr>
<tr>
<td>PPN</td>
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<td>( r = 0.3186 )</td>
<td>( r = 0.3889 )</td>
<td>( r = 0.3685 )</td>
</tr>
<tr>
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<td>( u = 3.6221 )</td>
<td>( u = 20.0167 )</td>
<td>( u = 22.414 )</td>
<td>( u = 4.1138 )</td>
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</table>

* \( r, u \) are dimensionless forms.
* \( r, r_{\text{min}} \) (miss distance)
* \( u \): \( |u|/l \) (energy)

Table 2. Comparisons of the performance of the four guidance laws \( (N_T = 5) \).

<table>
<thead>
<tr>
<th></th>
<th>Smart target</th>
<th>Step target</th>
<th>Ramp target</th>
<th>Sine Wave target</th>
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<td>HGL</td>
<td>( r = 0.3435 )</td>
<td>( r = 0.1248 )</td>
<td>( r = 0.1486 )</td>
<td>( r = 0.1114 )</td>
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<td>( u = 3.1549 )</td>
<td>( u = 2.739 )</td>
<td>( u = 0.8401 )</td>
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<td>( r = 0.2147 )</td>
<td>( r = 0.1847 )</td>
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<td>( u = 72.3149 )</td>
<td>( u = 240.2585 )</td>
<td>( u = 300.00 )</td>
<td>( u = 362.97 )</td>
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<td>TPN</td>
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<td>( r = 0.1799 )</td>
<td>( r = 0.1881 )</td>
<td>( r = 0.1798 )</td>
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<tr>
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<td>( u = 13.333 )</td>
<td>( u = 10.1116 )</td>
<td>( u = 36.166 )</td>
<td>( u = 40.123 )</td>
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<tr>
<td>PPN</td>
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<td>( r = 0.1805 )</td>
<td>( r = 0.1889 )</td>
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<td>( u = 12.9811 )</td>
<td>( u = 34.258 )</td>
<td>( u = 36.855 )</td>
</tr>
</tbody>
</table>

REFERENCES


Fig. 7. Miss distance v.s. weighting factor for HGL.

Fig. 8. Normalized cumulative velocity increment v.s. weighting factor for HGL.

Fig. 9. Trajectory of the phase plane.


