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Report on
“McKay’s correspondences for sporadic simple
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Abstract: This report has 3 sections. The first section is about the construction of
certain extension of unitary Virasoro VOAs which we discovered in our study of McKay’s
observation on the Monster group. We completely classified their irreducible modules
and computed their fusion rules. The second section is based on a joint work with M.
Miyamoto of University of Tsukuba and is about the 6-transposition property of the
Monster group and McKay’s $E_8$-observation. We show that certain subalgebras generated
by 2 conformal vectors of central charge 1/2 in the lattice VOA $V_{\sqrt{2}E_8}$ can be embedded
into the Moonshine VOA $V^\natural$. Thus, we establish a natural correspondence between the
numerical labels of the affine $E_8$-diagram and certain elements of the Monster group. Our
proof is based on the construction of the Leech lattice from some twistings of Niemeier
lattices. In particular, the Coxeter elements of the corresponding root system play an
important role in our argument.

The third section is a general structural theory of framed VOA, which is based on a
joint work with H. Yamauchi of University of Tokyo. The main motivation is to under-
stand the 4A-elements and 2-local structures of the Monster group. We show that the
structure codes $(C, D)$ of a framed VOA $V$ satisfy certain duality conditions. As a conse-
quence, we prove that every framed VOA is a simple current extension of the associated
binary code VOA $V_C$. This result would give a prospect on the classification of framed
vertex operator algebras. In addition, the pointwise frame stabilizer of $V$ is studied. We
completely determine all automorphisms in this pointwise stabilizer, which are of order 1,
2 or 4. The 4A-twisted sector and the 4A-twisted orbifold theory of the famous Moonshine
VOA \( V^z \) are also constructed explicitly. We verify that the top module of this twisted sector is of dimension 1 and of weight 3/4 and the VOA obtained by 4A-twisted orbifold construction of \( V^z \) is isomorphic to \( V^z \) itself.

1 Extension of tensor products of unitary Virasoro VOA

From the definition, a vertex operator algebra \( V \) contains a special element \( \omega \) such that the component operators \( L_n = \omega_{n+1}, n \in \mathbb{Z} \), will generate a Virasoro algebra inside \( V \). In other words, \( V \) admits a natural action of the Virasoro algebra \( Vir \). Conversely, Frenkel and Zhu [FZ] showed the irreducible highest weight module \( L(c, 0) \) of central charge \( c \) and highest weight 0 possesses a natural VOA structure. This class of VOA is called simple Virasoro VOAs. When the central charge \( c \) is given by

\[
c = c_m = 1 - \frac{6}{(m + 2)(m + 3)}, \quad m = 1, 2, \ldots,
\]

(1.1)

it is well-known [DM, FF, W] that \( L(c, 0) \) is rational and possesses an invariant unitary form. Moreover, the irreducible modules for \( L(c_m, 0) \) is given by \( L(c_m, h_{r,s}^m) \), where

\[
h_{r,s}^m = \frac{[r(m + 3) - s(m + 2)]^2 - 1}{4(m + 2)(m + 3)}
\]

(1.2)

and \( 1 \leq r \leq m + 1, 1 \leq s \leq m + 2 \).

Recently, there are many studies of the Moonshine module \( V^z \) using unitary Virasoro VOAs. Among them, Miyamoto and his Japanese group have constructed several interesting classes of vertex operator algebras using Virasoro VOAs [M1, KMY]. Moreover, they showed that many of them are embedded inside the Moonshine module \( V^z \) and their fusion rules can be used to define some explicit automorphisms of \( V^z \).

Partially motivated by their work, some simple current extension of the unitary Virasoro VOA \( L(c_m, 0) \) was studied in [LLY1]. It was shown that the vector space

\[
\tilde{V} = L(c_m, 0) \oplus L(c_m, h_{1,m+2}^m)
\]

has a simple VOA structure when \( m \equiv 0, 3 \mod 4 \) and a simple SVOA structure when \( m \equiv 1, 2 \mod 4 \). When \( \tilde{V} \) is a VOA, the set of all irreducible \( \tilde{V} \)-modules was completely classified and their fusion rules were determined in [LLY1].
We consider the case when $V^2$ is a SVOA (i.e., $m \equiv 1, 2 \pmod{4}$). More precisely, we will study the VOA

$$U = U^{m,n} = \mathcal{L}(c_m, 0) \otimes \mathcal{L}(c_n, 0) \oplus \mathcal{L}(c_m, h^{m}_{1,m+2}) \otimes \mathcal{L}(c_n, h^{n}_{1,n+2}),$$

for $m, n \equiv 1 \text{ or } 2 \pmod{4}$. As our main result, we will determine all irreducible modules for $U$. Moreover, the fusion rules among irreducible modules will be computed. Note that many of these VOAs are contained in the Moonshine module $V^2$. We believe that their fusion rules will be helpful in determining certain group theoretical properties of the Monster simple group.

1.1 Preliminaries

The method we used here is very similar to that in [LLY1]. In fact, most of the technical details has already be proved in [LLY1]. Now, let us review some basic facts from [LLY1]. Recall that the Virasoro algebra is the Lie algebra

$$Vir = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \bigoplus \mathbb{C} c$$

with the commutator relation

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{12} (m^3 - m) \delta_{m+n,0} c,$$

$$[L_m, c] = 0.$$

Let $L(c, h)$ be the irreducible highest module of $Vir$ of central charge $c$ and highest weight $h$, i.e.

$$L_n \cdot 1 = 0 \quad \text{for } n \geq 1,$$

$$L_0 \cdot 1 = h \cdot 1 \quad \text{and} \quad c \cdot 1 = c1.$$

It was shown by Frenkel and Zhu [FZ] that $L(c, 0)$ has a natural simple VOA structure. This class of VOAs is often referred to as simple Virasoro VOA.

Let

$$e_m = 1 - \frac{6}{(m+2)(m+3)} \quad (m = 1, 2, \ldots), \quad (1.3)$$
and
\[
h^{m}_{r,s} = \frac{r(m+3) - s(m+2))^2 - 1}{4(m+2)(m+3)},
\]
for \(r, s \in \mathbb{N}, 1 \leq r \leq m + 1, \) and \(1 \leq s \leq m + 2.\)

It is well-known [DM, FF, W] that \(L(c, m, 0), m = 1, 2 \ldots,\) are rational. Moreover, \(L(c, m, h^{m}_{r,s}), 1 \leq r \leq m + 1, 1 \leq s \leq m + 2,\) are exactly all the inequivalent irreducible modules of \(L(c, m, 0).\) The fusion rules for these modules are also known, which are given in [W] as follows (cf. [DM, FF]):

Let \(m \in \{1, 2, 3, \ldots\}.\) An ordered triple of pairs of integers \(((r, s), (r', s'), (r'', s''))\) is called \(\textit{admissible}\) if \(0 < r, r', r'' < m + 2, 0 < s, s', s'' < m + 3, r + r' + r'' < 2m + 4, s + s' + s'' < 2m + 6, r < r' + r'', r' < r + r'', r'' < r + r', s < s' + s'', s' < s + s'', s'' < s + s',\) and both \(r + r' + r''\) and \(s + s' + s''\) are odd. We will call a pairs of ordered triples \(((p, q), (p', q'), (p'', q''))\) \((r, s), (r', s'), (r'', s'')\) \(\textit{admissible}\) if both \(((p, q), (p', q'), (p'', q''))\) and \(((r, s), (r', s'), (r'', s''))\) are admissible.

\textbf{Theorem 1.1} (Wang [W]). \(\text{Let } m = 1, 2, \ldots \text{ and let}\)

\[
N^{(r,s)}_{(r',s'),(r'',s'')} = \dim I_{L(c, m, 0)}\left(L(c, m, h_{r,s}) L(c, m, h_{r',s'}) L(c, m, h_{r'',s''})\right),
\]

\[\text{where } I_{L(c, m, 0)}\left(M_{1}^{M_{2}} M_{3}\right) \text{ denotes the space of the intertwining operators of the type } (M_{1}^{M_{2}} M_{3}).\]

Then \(N^{(r,s)}_{(r',s'),(r'',s'')} \text{ is } 1 \text{ if and only if } ((r, s), (r', s'), (r'', s'')) \text{ is admissible; otherwise, }\)

\(N^{(r,s)}_{(r',s'),(r'',s'')} \text{ is equal to } 0.\)

The following facts can be obtained easily by definition.

\textbf{Lemma 1.2} (cf. [LLY1]). \(\text{For } m = 1, 2, 3, \ldots,\)

1. \(h^{m}_{r,s} = h^{m}_{m+2-r, m+3-s},\)

2. For any \((p, q) \neq (r, s) \text{ and } (p, q) \neq (m + 2 - r, m + 3 - s),\)

\[h^{m}_{r,s} \neq h^{m}_{p,q}.\]

3. For all \((r, s) \neq (1, m + 2) \text{ and } (r, s) \neq (m + 1, 1),\)

\[h^{m}_{1,m+2} > h^{m}_{r,s}.\]
In other words, \( h_{1,m+2}^m = h_{m+1,1}^m \) is the maximum among all the \( h_{r,n}^m \).

The following theorem is proved in [LLY1].

**Theorem 1.3.** Let \( h^m = h_{1,m+2}^m \) and \( \tilde{V} = \tilde{V}^m = L(c_m, 0) \oplus L(c_m, h^m) \). Then

1. \( \tilde{V} \) is a VOA if \( m \equiv 0 \) or \( 3 \mod 4 \),
2. \( \tilde{V} \) is a SVOA if \( m \equiv 1 \) or \( 2 \mod 4 \).

1.2 Extension of tensor product VOA and their irreducible modules

For now on, we will always assume that \( m, n \equiv 1 \) or \( 2 \mod 4 \). In this case, the modules \( \tilde{V}^m = L(c_m, 0) \oplus L(c_m, h^m) \) and \( \tilde{V}^n = L(c_n, 0) \oplus L(c_n, h^n) \) are both SVOAs. Hence the tensor product \( \tilde{V}^m \otimes \tilde{V}^n \) is a SVOA and their even part

\[
U = \tilde{U}^{m,n} = [L(c_m, 0) \otimes L(c_n, 0)] \oplus [L(c_m, h^m) \otimes L(c_n, h^n)]
\]

is a VOA.

In this paper, we will study the representation theory of \( U = \tilde{U}^{m,n} \). We will determine the irreducible modules for \( U \) and their fusion rules. In order to simplify the notation, we will denote the module \( L(c_m, h) \otimes L(c_n, k) \) by \( L^{m,n}(h \mid k) \) or simply by \( L(h \mid k) \) when there is no confusion.

Note that \( U \) is a simple current extension of \( L(c_m, 0) \otimes L(c_n, 0) \) and we will recall the following fact about simple current extension (cf. [DLM2, La, LLY1]).

**Theorem 1.4.** Let \( V \) be a simple rational VOA which satisfies Zhu’s \( C_2 \)-condition. Let \( M \) be a simple current module of \( V \) such that \( \tilde{V} = V + M \) has a VOA structure. Then as a \( V \)-module, an irreducible module of \( \tilde{V} \) must be of the following form:

1. \( N \oplus \tilde{N} \), where \( N \) is an irreducible \( V \)-module and \( \tilde{N} = M \times V N \not\equiv N \). In this case, \( N \oplus \tilde{N} \) has a unique \( \tilde{V} \)-module structure and the weight of \( \tilde{N} \) is congruent to the weight of \( N \mod \mathbb{Z} \).
2. \(N\), where \(N\) is an irreducible \(V\)-module such that \(N \cong M \times_V N\). In this case, there exist two non-isomorphic \(\tilde{V}\)-module structures on \(N\). We will denote these two modules by \(N^+\) and \(N^-\), respectively.

The modules in case (1) are often referred to as “non-split” type and the modules in case (2) are referred to as “split” type or \(\pm\)-type.

For a fixed positive odd integer \(n\), we will denote

\[
\tilde{h}^n_{r,s} = h^n_{r,n+3-s}, \quad \text{for all } r = 1, \ldots, n + 1 \text{ and } s = 1, \ldots, n + 2.
\]

Similarly, for \(n\) even, we will denote

\[
\tilde{h}^n_{r,s} = h^n_{n+2-r,s}, \quad \text{for all } r = 1, \ldots, n + 1 \text{ and } s = 1, \ldots, n + 2.
\]

Note that \(L(c_n, \tilde{h}^n_{r,s}) \cong L(c_n, h^n) \times L(c_n, h^n_{r,s})\).

In either case, denote

\[
\gamma_{p,q,r,s} = h^m_{p,q} + h^n_{r,s} - \tilde{h}^m_{p,q} - \tilde{h}^n_{r,s}
\]

and

\[
W^m_{(p,q),(r,s)} = W_{(p,q),(r,s)} = \left[ L\left(c_m, h^m_{p,q}\right) \otimes L\left(c_n, h^n_{r,s}\right) \right] \oplus \left[ L\left(c_m, \tilde{h}^m_{p,q}\right) \otimes L\left(c_n, \tilde{h}^n_{r,s}\right) \right]
\]

for any integers \(p, q, r, s\) such that \(1 \leq p < (m + 2), 1 \leq q < (m + 3), 1 \leq r < (n + 2)\) and \(1 \leq s < (n + 3)\).

**Lemma 1.5.** Let \(m\) and \(n\) be positive integers such that \(m, n \equiv 1 \mod 4\). Then for \(1 \leq p \leq (m + 2), 1 \leq q \leq (m + 3)/2, 1 \leq r \leq (n + 2)/2\) and \(1 \leq s \leq (n + 3)/2\), \(\gamma_{p,q,r,s}\) is an integer if and only if \(q + s\) is even. Moreover, \(\gamma_{p,q,r,s} = 0\) if and only if \(q = (m + 3)/2\) and \(s = (n + 3)/2\).

**Proof.** For any \(1 \leq p \leq (m + 2), 1 \leq q \leq (m + 3)/2, 1 \leq r \leq (n + 2)/2\) and \(1 \leq s \leq (n + 3)/2\),

\[
\gamma_{p,q,r,s} = -\frac{(m + 2 - 2p)(m + 3 - 2q) + (n + 2 - 2r)(n + 3 - 2s)}{4}. \quad (1.5)
\]
Since \( m, n \equiv 1 \mod 4 \), we have

\[
(m + 2 - 2p)(m + 3 - 2q) + (n + 2 - 2r)(n + 3 - 2s) \\
\equiv (m + 3 - (1 + 2p))(m + 3 - 2q) + (n + 3 - (1 + 2r))(n + 3 - 2s) \\
\equiv (4 - (1 + 2p))(4 - 2q) + (4 - (1 + 2r))(4 - 2s) \mod 4 \\
\equiv (1 + 2p)2q + (1 + 2r)2s \mod 4 \\
\equiv 2(q + s) \mod 4.
\]

Hence \( \gamma_{p,q,r,s} \) is an integer if and only if \( 2(q + s) \equiv 0 \mod 4 \). It is equivalent to \( q + s \equiv 0 \mod 2 \).

Hence, by Theorem 1.4 we have the following theorem.

**Theorem 1.6.** Let \( m, n \equiv 1 \mod 4 \) and \( W \) an irreducible \( U^{m,n} \)-module. Then as an \( L(0 \mid 0) \)-module, either

1. \( W \cong W_{(p,q),(r,s)} \) for some \( p, q, r, s \) such that \( 1 \leq p < (m + 2), 1 \leq q < (m + 3)/2, 1 \leq r < (n + 2)/2, 1 \leq s < (n + 3)/2 \) and \( q + s \) is even, or

2. \( W \cong L \left( h_{p,(m+3)/2}^m \mid h_{r,(n+3)/2}^n \right) \) for \( 1 \leq p < (m + 2) \) and \( 1 \leq r < (n + 2)/2 \). In this case, there are two non-isomorphic \( U \)-module structures. We will denote them by \( L \left( h_{p,(m+3)/2}^m \mid h_{r,(n+3)/2}^n \right)^\pm \).

For the case \( m \equiv 1 \mod 4 \) and \( n \equiv 2 \mod 4 \), we can also get the following similar results.

**Lemma 1.7.** Let \( m \) and \( n \) be positive integers such that \( m \equiv 1 \mod 4 \) and \( n \equiv 2 \mod 4 \). Then for \( 1 \leq p \leq (m + 2), 1 \leq q \leq (m + 3)/2, 1 \leq r \leq (n + 2)/2 \) and \( 1 \leq s \leq (n + 3)/2 \), \( \gamma_{p,q,r,s} \) is an integer if and only if \( q + r \) is even. Moreover, \( \gamma_{p,q,r,s} = 0 \) if and only if \( q = (m + 3)/2 \) and \( r = (n + 2)/2 \).

**Theorem 1.8.** Let \( m \equiv 1 \mod 4 \) and \( n \equiv 2 \mod 4 \) and \( W \) an irreducible \( U^{m,n} \)-module. Then as an \( L(0 \mid 0) \)-module, either
1. \( W \cong W_{(p,q),(r,s)} \) for some \( p, q, r, s \) such that \( 1 \leq p < (m+2)/2, 1 \leq q < (m+3)/2, 1 \leq r < (n+2)/2, 1 \leq s < (n+3)/2 \) and \( p + r \) is even, or

2. \( W \cong L \left( \frac{h^m_{p,(m+3)/2}}{h^n_{(n+2)/2,s}} \right) \) for \( 1 \leq p < (m+2) \) and \( 1 \leq s < (n+3)/2 \). In this case, there are two non-isomorphic \( U \)-module structures. We will denote them by \( L \left( \frac{h^m_{p,(m+3)/2}}{h^n_{(n+2)/2,s}} \right)^\pm \).

We also can obtain the following similar results for the case \( m, n \equiv 2 \) mod 4.

**Lemma 1.9.** Let \( m \) and \( n \) be positive integers such that \( m, n \equiv 2 \) mod 4. Then for
\[
1 \leq p \leq (m+2)/2, 1 \leq q \leq (m+3), 1 \leq r \leq (n+2)/2 \text{ and } 1 \leq s \leq (n+3)/2,
\]
\( \gamma_{p,q,r,s} \) is an integer if and only if \( p + r \) is even. Moreover, \( \gamma_{p,q,r,s} = 0 \) if and only if \( p = (m+2)/2 \) and \( r = (n+2)/2 \).

**Theorem 1.10.** Let \( m, n \equiv 2 \) mod 4 and \( W \) an irreducible \( U^m,n \)-module. Then as an \( L(0 \mid 0) \)-module, either

1. \( W \cong W_{(p,q),(r,s)} \) for some \( p, q, r, s \) such that \( 1 \leq p < (m+2)/2, 1 \leq q < (m+3), 1 \leq r < (n+2)/2 \text{ and } 1 \leq s < (n+3)/2, \) and \( p + r \) is even, or

2. \( W \cong L \left( \frac{h^m_{(m+2)/2,q}}{h^n_{(n+2)/2,s}} \right) \) for \( 1 \leq q < (m+3) \) and \( 1 \leq s < (n+3)/2 \). In this case, there are two non-isomorphic \( U \)-module structures. We will denote them by \( L \left( \frac{h^m_{(m+2)/2,q}}{h^n_{(n+2)/2,s}} \right)^\pm \).

### 1.3 Fusion rules

In this subsection, we will determine the fusion rules for irreducible \( U \)-modules when \( m, n \equiv 1 \) or 2 mod 4. The method is essentially the same as [LLY1]. The main tool is the following proposition by Dong and Lepowsky [DL].

**Proposition 1.11** (Proposition 11.9 of Dong-Lepowsky [DL]). Let \( W^1, W^2 \) and \( W^3 \) be \( V \)-modules and let \( I \) be an intertwining operator of type

\[
\begin{pmatrix}
W^3 \\
W^1 & W^2
\end{pmatrix}
\]
Assume that $W^1$ and $W^2$ have no proper submodules containing $v^1$ and $v^2$ respectively. Then $I(v^1, z)v^2 = 0$ implies $I(\cdot, z) = 0$.

Now let us fix our notations. Let $U_0 = L(c_m, 0) \otimes L(c_n, 0)$ and $U_1 = L(c_m, h^m_{1,m+2}) \otimes L(c_n, h^n_{1,n+2})$. Then $U = U^{m,n} = U_0 \oplus U_1$.

For any $1 \leq p, p', p'' \leq m + 2$, $1 \leq r, r', r'' \leq n + 2$, $1 \leq q, q', q'' \leq m + 3$ and $1 \leq s, s', s'' \leq n + 3$, let

$$M_0 = L(h^m_{p,q} | h^n_{r,s}), \quad N_0 = L(h^m_{p',q'} | h^n_{r',s'}), \quad \text{and} \quad L_0 = L(h^m_{p'',q''} | h^n_{r'',s''}).$$

Moreover, we denote

$$M_1 = U_1 \times M_0 = L(h^m_{p,q} | h^n_{r,s}), \quad N_1 = U_1 \times N_0 = L(h^m_{p',q'} | h^n_{r',s'}), \quad \text{and} \quad L_1 = U_1 \times L_0 = L(h^m_{p'',q''} | h^n_{r'',s''}).$$

We will also assume that $M_0 \oplus M_1$, $N_0 \oplus N_1$, and $L_0 \oplus L_1$ have $U$-modules structure. Without loss, we may also fix the $U$-module structure on $M_0 \oplus M_1$, $N_0 \oplus N_1$, and $L_0 \oplus L_1$ such that

$$Y(a, z)m \in M_1[[z, z^{-1}]], \quad Y(a, z)n \in N_1[[z, z^{-1}]] \quad \text{and} \quad Y(a, z)b \in L_1[[z, z^{-1}]]$$

for any $a \in V_1$, $m \in M_0$, $n \in N_0$ and $b \in L_0$. Note that if $M_0 \cong M_1 \cong M$, then $M_0 \oplus M_1 \cong M^+ \oplus M^-$ as $U$-modules.

Next we will consider intertwining operators of the type

$$I_U \left( \begin{array}{c} L_0 \oplus L_1 \\ M_0 \oplus M_1 \\ N_0 \oplus N_1 \end{array} \right).$$

**Lemma 1.12.** Let $M_0, N_0, L_0$, and $M_1, N_1, L_1$ be defined as above. Then,

$$\dim I_U \left( \begin{array}{c} L_0 \oplus L_1 \\ M_0 \oplus M_1 \\ N_0 \oplus N_1 \end{array} \right) \leq \dim I_{U_0} \left( \begin{array}{c} L_1 \\ M_0 \oplus N_0 \end{array} \right) + \dim I_{U_0} \left( \begin{array}{c} L_1 \\ M_0 \oplus N_0 \end{array} \right) \leq 2.$$
Proof. By Proposition 11.9 of [DL], it is clear that
\[
\dim I_U \left( \begin{array}{c@{\oplus}c} L_0 & L_1 \\ M_0 & N_0 \end{array} \right) \\
\leq \dim I_{U_0} \begin{pmatrix} L_0 & L_1 \\ M_0 & N_0 \end{pmatrix} \\
= \dim I_{U_0} \left( \begin{array}{c} L_0 \\ M_0 \end{array} \right) + \dim I_{U_0} \left( \begin{array}{c} L_1 \\ N_0 \end{array} \right).
\]
Thus, we have the desired result. \qed

Suppose \( I_V (M_0 \begin{array}{c} L_0 \\ N_0 \end{array}) \neq 0 \). By using the admissible conditions, it is easy to see that \( I_V (M_1 \begin{array}{c} L_1 \\ N_0 \end{array}) \), \( I_V (M_0 \begin{array}{c} L_1 \\ N_1 \end{array}) \) and \( I_V (M_1 \begin{array}{c} L_0 \\ N_1 \end{array}) \) are also non-zero. By the same proof as [LLY1, Lemma 5.3], we also have the following lemma.

Lemma 1.13. Let \( I \) be a nonzero \( V \)-intertwining operator of the type \( I_V (M_0 \begin{array}{c} L_0 \\ N_0 \end{array}) \). Then, there is a nonzero \( U \)-intertwining operator \( Y \) of the type
\[
I_U \left( \begin{array}{c@{\oplus}c} L_0 & L_1 \\ M_0 & N_0 \end{array} \right)
\]
such that
\[
Y(m_0, z)n_0 = I(m_0, z)n_0 \quad \text{for any} \quad m_0 \in M_0, n_0 \in N_0.
\]

Remark 1.14. By the lemma, we actually have
\[
\dim I_U \left( \begin{array}{c@{\oplus}c} L_0 & L_1 \\ M_0 & N_0 \end{array} \right) \geq \dim I_{U_0} \left( \begin{array}{c} L_0 \\ M_0 \end{array} \right) + \dim I_{U_0} \left( \begin{array}{c} L_1 \\ N_0 \end{array} \right)
\]
and hence
\[
\dim I_U \left( \begin{array}{c@{\oplus}c} L_0 & L_1 \\ M_0 & N_0 \end{array} \right) = \dim I_{U_0} \left( \begin{array}{c} L_0 \\ M_0 \end{array} \right) + \dim I_{U_0} \left( \begin{array}{c} L_1 \\ N_0 \end{array} \right).
\]
As a corollary, we have
Theorem 1.15. (1) If $M_0 \not\equiv M_1$, $N_0 \not\equiv N_1$ and $L_0 \not\equiv L_1$, then

$$
\dim I_U \left( \frac{L_0 + L_1}{M_0 + M_1} \frac{N_0 + N_1}{L_0 + N_0} \right) = \begin{cases} 
0, & \text{if } I_U \left( \frac{L_0}{M_0} \frac{N_0}{L_0 + N_0} \right) = I_U \left( \frac{L_1}{M_0} \frac{N_0}{L_0 + N_0} \right) = 0, \\
1, & \text{if } I_U \left( \frac{L_0}{M_0} \frac{N_0}{L_0 + N_0} \right) \neq 0 \text{ and } I_U \left( \frac{L_1}{M_0} \frac{N_0}{L_0 + N_0} \right) = 0, \\
2, & \text{if } I_U \left( \frac{L_0}{M_0} \frac{N_0}{L_0 + N_0} \right) \neq 0 \text{ and } I_U \left( \frac{L_1}{M_0} \frac{N_0}{L_0 + N_0} \right) \neq 0.
\end{cases}
$$

(2) If $M_0 \not\equiv M_1$, $N_0 \not\equiv N_1$, and $L \cong V_1 \times L$, then

$$
\dim I_U \left( \frac{L^\pm}{N_0 + N_1} \frac{M_0 + M_1}{L^\pm} \right) = \dim I_U \left( \frac{L_0}{M_0 + M_1} \frac{N_0}{N_0 + N_1} \right) = 1
$$

if and only if $I_U \left( \frac{L}{N_0} \frac{M_0}{N_0 + N_1} \right) \neq 0$.

Next, we will study the fusion rules involving two or more modules of $\pm$-type. The method is again similar to [LLY1]. In order to treat all cases at once, we will adopt the following notation.

For any $m = 1, 2, \ldots$, $1 \leq r < m + 2$, and $1 \leq s < m + 3$, we denote

$$(r, m) = \begin{cases} 
(r, (m + 3)/2), & \text{if } n \equiv 1 \mod 4, \\
((m + 2)/2, r), & \text{if } n \equiv 2 \mod 4,
\end{cases}$$

$$(r, s) = \begin{cases} 
(r, m + 3 - s), & \text{if } n \equiv 1 \mod 4, \\
((m + 2 - r, s), & \text{if } n \equiv 2 \mod 4.
\end{cases}$$

Note that $\tilde{h}_{r,s}^m = h_{r,s}^m$ and $\tilde{h}_{r,s}^m = h_{r,s}^m$. Now denote

$$M_{i,j} = M_{i,j}^{m,n} = L \left( h_{i,m}^m \bigg| h_{j,m}^m \right),$$

for $1 \leq i < (m + 3)$ and $1 \leq j < (n + 3)/2$. In other words,

$$M_{i,j} = \begin{cases} 
L \left( h_{i,(m+3)/2}^m \bigg| h_{j,(n+3)/2}^n \right), & \text{if } m \equiv 1 \mod 4 \text{ and } n \equiv 1 \mod 4, \\
L \left( h_{i,(m+3)/2}^m \bigg| h_{n+2/2,j}^n \right), & \text{if } m \equiv 1 \mod 4 \text{ and } n \equiv 2 \mod 4, \\
L \left( h_{(m+2)/2,i}^m \bigg| h_{(n+2)/2,j}^n \right), & \text{if } m \equiv 2 \mod 4 \text{ and } n \equiv 2 \mod 4,
\end{cases}$$

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Next we will recall the notation of contragredient (dual) module. Let \( N = \coprod_{n \in \mathbb{Q}} N_n \) be a \( U \)-module. The contragredient module \( N^* = \coprod_{n \in \mathbb{Q}} N^*_n \) is defined to be the graded dual space of \( N \) with the adjoint vertex operator \( Y^*(v, z) \) determined by

\[
\langle Y^*(a, z) b', u \rangle = \left\langle b', Y \left( e^{zL_1} \left( -z^{-2} \right)^L_0 a, z^{-1} \right) u \right\rangle.
\]

**Theorem 1.16.** Let \( m, n \equiv 1 \) or \( 2 \mod 4 \). Then

1. \((M_{p, r})^+ \cong M_{p, r}^+\) and \((M_{p, r}^-)^* \cong M_{p, r}^-\) as \( U \)-modules when \( m + n \equiv 0, 3, 6 \mod 8 \),

2. \((M_{p, r})^+ \cong M_{p, r}^-\) and \((M_{p, r}^-)^* \cong M_{p, r}^+\) as \( U \)-modules when \( m + n \equiv 2, 4, 7 \mod 8 \).

**Proof.** Let \((M_{p, r}^*, Y)\) be the irreducible \( \tilde{V} \)-module \( M_{p, r}^+ \) or \( M_{p, r}^- \). First, we will note that \( M_{p, r} \cong M_{p, r}^* \) as a \( L(0 \mid 0) \)-module. Thus we can identify the vector spaces of \( M_{p, r} \) and \( M_{p, r}^* \) by using the \( L(0 \mid 0) \)-contragredient form on \( M_{p, r} \) and note that this form is symmetry(cf. [FHL, Li]). On the other hand, \( M_{p, r} \) and \((M_{p, r}^*)^*\) are isomorphic to \( M_{p, r}^* \) and \( M_{p, r}^* \) as \( L(0 \mid 0) \)-modules. Hence we can identify the vector spaces of \( M_{p, r}^* \) and \((M_{p, r}^*)^*\) also. Moreover, we may assume the action of \( L(c_m, 0) \) on both \( M_{p, r}^* \) and \((M_{p, r}^*)^*\) are exactly the same. By abuse of notation, we will denote \( M_{p, r} \) and \((M_{p, r}^*)^*\) by \( M_{p, r} \) when only the vector space (or \( L(0 \mid 0) \)-module) structure is considered.

Now let \( v \) be a highest weight vector of weight \( h_m + h_n \) in \( L(h_m \mid h_n) \). Then for any \( b, b' \in M_{p, r} \),

\[
\langle Y^*(v, z) b', b \rangle = \left\langle b', Y \left( e^{zL_1} \left( -z^{-2} \right)^L_0 v, z^{-1} \right) b \right\rangle = \left\langle b', Y \left( \left\langle -z^{-2} \right\rangle^{h_m + h_n} v, z^{-1} \right) b \right\rangle.
\]

Thus, for any \( k \in \mathbb{Z} \),

\[
\langle v_k^* b', b \rangle = \langle b', (-1)^{(h_m + h_n)} v_{-k+2(h_m + h_n)-2} b \rangle,
\]

where \( Y^*(v, z) = \sum_{k \in \mathbb{Z}} v_k^* z^{-k-1} \).

Let \( k = h_m + h_n - 1 \) and let \( b = b' \) be a highest weight vector of \( M_{p, r} \), which is unique up to a scalar. Then

\[
\langle v_{h_m + h_n - 1}^* b, b \rangle = \langle b, (-1)^{(h_m + h_n)} v_{h_m + h_n - 1} b \rangle.
\]
and we have
\[ v_{h^m + h^n - 1}^* b = (-1)^{(h^m + h^n)} v_{h^m + h^n - 1} b. \quad (1.6) \]
Note that \( v_{h^m + h^n - 1} b \) and \( \langle b, b \rangle \) are nonzero and both \( v_{h^m + h^n - 1} b \) and \( v_{h^m + h^n - 1}^* b \) are multiples of \( b \).

Since both \( Y^*(a, z)b \) and \( Y(a, z)b \), for \( a \in U_1 = L(h^m | h^n) \) and \( b \in M_{p,r} \), are both \( L(0 | 0) \)-intertwining operators of the type \( I(U_1^{M_{p,r}} M_{p,r}) \),
\[ Y^*(a, z)b = \lambda Y(a, z)b \quad \text{for some } \lambda \in \mathbb{C} \]
and thus by (1.6), we have \( Y^*(a, z)b = (-1)^{(h^m + h^n)} Y(a, z)b \) for any \( a \in U_1 \) and \( b \in M_{p,r} \). Since \( h^m + h^n = \frac{m(m+1)+n(n+1)}{4} \) is even when \( m + n \equiv 0, 3, 6 \mod 8 \) and is odd when \( m \equiv 2, 4, 7 \mod 8 \), we have
\[ Y^*(a, z)b = \begin{cases} 
Y(a, z)b, & \text{if } m + n \equiv 0, 3, 6 \mod 8, \\
-Y(a, z)b, & \text{if } m + n \equiv 2, 4, 7 \mod 8.
\end{cases} \]
Therefore, if \( m + n \equiv 0, 3, 6 \mod 8 \), \((M^\epsilon_{p,r})^* \cong M^\epsilon_{p,r}\) but if \( m + n \equiv 2, 4, 7 \mod 8 \), \((M^\epsilon_{p,r})^* \not\cong M^\epsilon_{p,r}\) and thus \((M^+_{p,r})^* \cong M^-_{p,r}\) and \((M^-_{p,r})^* \cong M^+_{p,r}\). □

**Theorem 1.17.** Let \( M = M_{1,1} \). Then
\[ I_U \begin{pmatrix} (M^\epsilon)^* \\ M^\epsilon_1 \\ M^\epsilon_2 \end{pmatrix} \neq 0 \]
if and only if \( \epsilon = \epsilon_1 = \epsilon_2 \).

**Proof.** By the \( S_3 \)-symmetry of fusion rules (cf. [FHL, Li]), we have
\[ \dim I_U \begin{pmatrix} (M^\epsilon)^* \\ M^\epsilon \end{pmatrix} = \dim I_U \begin{pmatrix} (M^\epsilon)^* \\ M^- \end{pmatrix} = \dim I_U \begin{pmatrix} (M^-)^* \\ M^\epsilon \end{pmatrix}. \]
Now suppose
\[ I_U \begin{pmatrix} (M^-)^* \\ M^\epsilon \end{pmatrix} \neq 0. \]
Then,
\[ I_U \begin{pmatrix} M^+ \oplus M^- \\ M^+ \oplus M^- \end{pmatrix} \geq 3. \]
It is impossible by Lemma 1.12. Hence,

\[ I_U \left( \begin{pmatrix} (M^-)^* \\ M^+ \end{pmatrix} \right) = I_U \left( \begin{pmatrix} (M^+)^* \\ M^- \end{pmatrix} \right) = 0 \]

and

\[ I_U \left( \begin{pmatrix} (M^-)^* \\ M^+ \end{pmatrix} \right) = I_U \left( \begin{pmatrix} (M^-)^* \\ M^- \end{pmatrix} \right) = I_U \left( \begin{pmatrix} (M^-)^* \\ M^+ \end{pmatrix} \right) = 0. \]

Hence

\[ I_U \left( \begin{pmatrix} (M^+)^* \\ M^- \end{pmatrix} \right) = I_U \left( \begin{pmatrix} (M^-)^* \\ M^- \end{pmatrix} \right) = 1 \]

since

\[ I_U \left( \begin{pmatrix} M^+ \oplus M^- \\ M^+ \oplus M^- \end{pmatrix} \right) = 2. \]

Now let us denote

\[ [p, q]_m = \begin{cases} (p, q) & \text{if } m \text{ is odd}, \\ (q, p) & \text{if } m \text{ is even}, \end{cases} \]

Theorem 1.18. Let \( m, n \equiv 1 \) or \( 2 \) mod 4 and denote

\[ W(p, r) = W[p,1]_{m \cdot [r,1]} = \begin{cases} W(p,1), & \text{if } m \equiv 1 \text{ mod 4 and } n \equiv 1 \text{ mod 4}, \\ W(p,1), & \text{if } m \equiv 1 \text{ mod 4 and } n \equiv 2 \text{ mod 4}, \\ W(1,p), & \text{if } m \equiv 2 \text{ mod 4 and } n \equiv 2 \text{ mod 4}. \end{cases} \]

Then we have

\[ \dim I_U \left( \begin{pmatrix} M^\epsilon_{p,r} \\ W(p,r) \end{pmatrix} M_{1,1}^\epsilon \right) + \dim I_U \left( \begin{pmatrix} M^\epsilon_{p,r} \\ W(p,r) \end{pmatrix} M_{1,1}^\epsilon \right) = 1, \]

where \( \epsilon = \pm 1 \) and we identify the integers \( +1 \) as “+” and \( -1 \) as “−”.

Proof. We will only prove the case for \( m \equiv 1 \) mod 4 and \( n \equiv 1 \) mod 4. The other two cases are similar.

By Proposition 1.11 (cf. [DL, Proposition 11.9]) and the module structures of \( M^+ \) and \( M^- \), we have

\[ \dim I_U \left( \begin{pmatrix} M^\epsilon_{p,r} \\ W_{p,1}(r,1) \end{pmatrix} M_{1,1}^\epsilon \right) + \dim I_U \left( \begin{pmatrix} M^\epsilon_{p,r} \\ W_{p,1}(r,1) \end{pmatrix} M_{1,1}^\epsilon \right) \leq 1 \]
and
\[ \dim I_U \begin{pmatrix} M_{p,r}^\epsilon & M_{1,1}^- \end{pmatrix} + \dim I_U \begin{pmatrix} M_{p,r}^- & M_{1,1}^\epsilon \end{pmatrix} \leq 1. \]

Now by the admissible condition and Lemma 1.13,
\[ \dim I_U \begin{pmatrix} M_{p,r}^\epsilon + M_{p,r}^- & M_{1,1}^\epsilon + M_{1,1}^- \end{pmatrix} = 2. \]

Hence
\[ \dim I_U \begin{pmatrix} M_{p,r}^\epsilon & M_{1,1}^- \end{pmatrix} + \dim I_U \begin{pmatrix} M_{p,r}^- & M_{1,1}^\epsilon \end{pmatrix} = 1 \]
and
\[ \dim I_U \begin{pmatrix} M_{p,r}^\epsilon & M_{1,1}^- \end{pmatrix} + \dim I_U \begin{pmatrix} M_{p,r}^- & M_{1,1}^\epsilon \end{pmatrix} = 1. \]

By the above theorem, we can define \( L \left( \frac{h_{p,(m+3)/2}}{h_{r,(n+3)/2}} \right)^\epsilon \) as follows:

**Definition 1.19.** Suppose that \( M_{1,1}^\pm \) are defined for \( m, n \equiv 1 \) or \( 2 \) mod 4. Then \( M_{p,r}^+ \) and \( M_{p,r}^- \) are both defined to be the unique irreducible \( U \)-module of highest weight
\[
\begin{cases} 
\frac{h_{p,(m+3)/2}}{h_{r,(n+3)/2}}, & \text{if } m \equiv 1 \text{ mod } 4 \text{ and } n \equiv 1 \text{ mod } 4 \\
\frac{h_{p,(m+3)/2}}{h_{n,(n+2)/2}}, & \text{if } m \equiv 1 \text{ mod } 4 \text{ and } n \equiv 2 \text{ mod } 4 \\
\frac{h_{m,(m+2)/2}}{h_{n,(n+2)/2}}, & \text{if } m \equiv 2 \text{ mod } 4 \text{ and } n \equiv 2 \text{ mod } 4
\end{cases}
\]
such that
\[ \dim I_U \left( \frac{M_{p,r}^+}{W(p, r)} \frac{M_{1,1}^+}{M_{1,1}} \right) = 1 \]
and
\[ \dim I_U \left( \frac{M_{p,r}^-}{W(p, r)} \frac{M_{1,1}^-}{M_{1,1}} \right) = 0. \]

**Remark 1.20.** Note that we also have
\[ \dim I_U \left( \frac{M_{p,r}^-}{W(p, r)} \frac{M_{1,1}^-}{M_{1,1}} \right) = 1 \]
and
\[ \dim I_U \left( \frac{M_{p,r}^+}{W(p, r)} \frac{M_{1,1}^+}{M_{1,1}} \right) = 0. \]
Theorem 1.21. Let $m, n \equiv 1$ or 2 mod 4. Then

$$I_U = \begin{pmatrix} M_{1,1}^{c^2} & M_{1,1}^{c^1} \\ W_{[1,q,m],[1,s,n]} & M_{1,1}^{c^1} \end{pmatrix} \neq 0$$

if and only if

$$(-1)^{(q+s)/2} = -\epsilon_1 \epsilon_2, \quad \text{if } m \equiv 1 \text{ mod 4 and } n \equiv 1 \text{ mod 4},$$

$$(-1)^{(q-s)/2} = \epsilon_1 \epsilon_2, \quad \text{if } m \equiv 1 \text{ mod 4 and } n \equiv 2 \text{ mod 4},$$

$$(-1)^{(q+s)/2} = -\epsilon_1 \epsilon_2, \quad \text{if } m \equiv 2 \text{ mod 4 and } n \equiv 2 \text{ mod 4}.$$ 

Proof. We only prove the first case $m \equiv 1 \text{ mod 4 and } n \equiv 1 \text{ mod 4}$. The proofs of the other two cases are similar.

Consider $W_{[1,q,m],[1,s,n]} = W_{1,q),(1,s)} = L(h_{1,q}^m \ | \ h_{1,s}^n) \oplus L(\tilde{h}_{1,q}^m \ | \ \tilde{h}_{1,s}^n)$ for $1 \leq q < (m + 3), 1 \leq s < (n + 3)/2$. Since $m \equiv 1 \text{ mod 4 and } n \equiv 1 \text{ mod 4},$ if $q + s$ is even, then by Lemma 1.5

$$\gamma_{1,q,1,s} = -\frac{m(m + 3 - 2q) + n(n + 3 - 2s)}{4}$$

$$= -\frac{m(m + 1) + n(n + 1)}{4} + \frac{m(q - 1) + n(s - 1)}{2}$$

$$= -\frac{m(m + 1) + n(n + 1)}{4} + \frac{(m - 1)(q - 1) + (n - 1)(s - 1)}{2} + \frac{(q + s)}{2} - 1 \quad (1.7)$$

is an integer. Let $\epsilon = \pm 1$. By the same argument in Theorem 1.18, we know that

$$\dim I_U \begin{pmatrix} M_{1,1}^c & M_{1,1}^{c^1} \\ W_{(1,q),(1,s)} & M_{1,1}^{c^1} \end{pmatrix} + \dim I_U \begin{pmatrix} M_{1,1}^{-c} & M_{1,1}^{-c^1} \\ W_{(1,q),(1,s)} & M_{1,1}^{-c^1} \end{pmatrix} = 1.$$

Now suppose there exists a nonzero $U$ intertwining operator

$$\mathcal{V} \in I_U \begin{pmatrix} M_{1,1}^c & M_{1,1}^{c^1} \\ W_{(1,q),(1,s)} & M_{1,1}^{c^1} \end{pmatrix}.$$ 

Then the contragredient intertwining operator

$$\mathcal{V}^* \in I_U \begin{pmatrix} (M_{1,1}^c)^* & (M_{1,1}^{c^1})^* \\ W_{(1,q),(1,s)} & (M_{1,1}^{c^1})^* \end{pmatrix},$$

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where $\mathcal{Y}^*$ is determined by
\[
\langle \mathcal{Y}^*(u, z)v^*, w \rangle = \langle v^*, \mathcal{Y}(e^{zL_1}(e^{x_1z-2})L_0 u, z^{-1})w \rangle,
\]
for $u \in W_{(1,q), (1,s)}$, $v^* \in (M^*_1)^*$, and $w \in M^*_1$.

Let $u^0$ be a highest weight vector of $L(h^m_{1,q} \mid h^n_{1,s})$ and $u^1$ be a highest weight vector of $L\left(\tilde{h}^m_{1,q} \mid \tilde{h}^n_{1,s}\right)$ and let $a$ be a highest weight vector of $L(h^m \mid h^n)$ such that
\[
u^1 = a_{vta-1} wta^0 u^0.
\]
Let $w$ be a highest weight vector of $M^*_1$, and $v^*$ be a highest weight vector of $(M^*_1)^*$ and let $\phi : M^*_1 \rightarrow (M^*_1)^*$ be a $V$-isomorphism such that $\phi w = v^*$.

Then by the proof of Theorem 1.16, $\phi \mathcal{Y}(a, z) = (-1)^{[m(m+1)+n(n+1)]/4} \mathcal{Y}(a, z)\phi$. Moreover, we have $\mathcal{Y}^*(u, z)\phi = \lambda \phi \mathcal{Y}(u, z)$ for any $u \in L(h^m_{1,q} \mid h^n_{1,s})$ and for some $\lambda \neq 0$ since both $\phi \mathcal{Y}(u, z)v$ and $\mathcal{Y}^*(u, z)\phi v$, for $u \in L(h^m_{1,q} \mid h^n_{1,s})$ and $v \in M^*_1$, are $L(0 \mid 0)$-intertwining operators of type
\[
\begin{pmatrix}
\mathcal{Y}_{1,1} &
\mathcal{Y}(h^m_{1,q} \mid h^n_{1,s}) &
\mathcal{Y}_{1,1}
\end{pmatrix}.
\]

Since $\mathcal{Y} \neq 0$, there exist nonzero constants $\alpha, \beta$ such that $o(u^0)w = \alpha w$ and $o(u^1)w = \beta w$. Thus, we have
\[
\alpha \langle v^*, w \rangle = \langle v^*, o(u^0)w \rangle = e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \langle o^*(u^0)v^*, w \rangle = e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \lambda \alpha \langle v^*, w \rangle,
\]
\[
\beta \langle v^*, w \rangle = \langle v^*, o(u^1)w \rangle = e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \langle o^*(u^1)v^*, w \rangle
\]
\[
= e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \langle o^*(a_{vta-1} wta^0 - u^0)\phi w, w \rangle
\]
\[
= e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \lambda (-1)^{[m(m+1)+n(n+1)]/4} \beta \langle v^*, w \rangle.
\]

Therefore, we have
\[
\alpha = e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \lambda \alpha,
\]
\[
\beta = e^{-(h^m_{1,q} + h^n_{1,s})\pi i} \lambda (-1)^{[m(m+1)+n(n+1)]/4} \beta.
\]
and hence by (1.7)

\[ 1 = e^{(\tilde{h}_m^{m_1} + \tilde{h}_n^{n_1}) - (\tilde{h}_m^{m_1} + \tilde{h}_n^{n_1})} \pi \cdot (-1)^{m(m+1)+n(n+1)/4} \]

\[ = (-1)^{(q,1) + (n,1)} \cdot (-1)^{m(m+1)+n(n+1)/4} \]

\[ = (-1)^{(m-1)(q-1)+n(n-1)/2+(q+s)/2-1}. \]

Since \( m \equiv 1 \mod 4 \) and \( n \equiv 1 \mod 4 \), it is impossible if \((q + s)/2\) is even.

Therefore

\[ I_U \left( \begin{array}{c} M_{1,1}^\epsilon \\ W_{(1,q), (1,s)} M_{1,1}^\epsilon \end{array} \right) = 0 \]

and hence

\[ I_U \left( \begin{array}{c} M_{1,1}^{-\epsilon} \\ W_{(1,q), (1,s)} M_{1,1}^\epsilon \end{array} \right) \neq 0, \]

if \((q + s)/2\) is even.

Next let \( \mathcal{Y} \in I_U \left( \begin{array}{c} M_{1,1}^{-\epsilon} \\ W_{(1,q), (1,s)} M_{1,1}^\epsilon \end{array} \right) \) be a nonzero intertwining operator and then

the contragredient operator \( \mathcal{Y}^* \in I_U \left( \begin{array}{c} (M_{1,1}^\epsilon)^* \\ W_{(1,q), (1,s)} (M_{1,1}^{-\epsilon})^* \end{array} \right) \).

Let \( w^\pm \) be highest weight vectors of \( M_{1,1}^\pm \) and \((v^\pm)^*\) highest vectors of \((M_{1,1}^\pm)^*\). Let \( \phi^\pm : M_{1,1}^\pm \to (M_{1,1}^\pm)^* \) be \(V\)-isomorphisms such that \( \phi^\pm w^\pm = (v^\pm)^* \). Since both \( \phi^- \mathcal{Y}(u, z) \) and \( \mathcal{Y}^*(u, z)\phi^+ w^+ \), for \( u \in L(h_m^m \mid h_n^n) \) and \( w \in M_{1,1}^\pm \), are \(L(0 \mid 0)\)-intertwining operators of type

\[ \left( \begin{array}{c} M_{1,1}^\pm \\ L(h_m^m \mid h_n^n) M_{1,1}^\pm \end{array} \right), \]

there exists \( \mu \neq 0 \) such that \( \mu \phi^- \mathcal{Y}(u, z) = \mathcal{Y}^*(u, z)\phi^+ \). Note also that \( Y^*(a, z)\phi^+ = (-1)^{m(m+1)+n(n+1)/2+1}\phi^+ Y(a, z) \) for \( a \in L(h_m^m \mid h_n^n) \). Moreover, there exist nonzero con-
stants $\alpha$, $\beta$ such that $o(u^0)w^+ = \alpha w^-$ and $o(u^1)w^+ = \beta w^-$. Thus, we have

\[
\alpha((v^-)^*, w^-) = \langle (v^-)^*, o(u^0)w^+ \rangle \\
= e^{-(h_{1,q}^m + h_{1,s}^n)\pi i} \langle o^*(u^0)(v^-)^*, w^+ \rangle \\
= e^{-(h_{1,q}^m + h_{1,s}^n)\pi i} \mu \alpha ((v^+)^*, w^+),
\]

\[
\beta((v^-)^*, w^-) = \langle (v^-)^*, o(u^1)w^+ \rangle \\
= e^{-(\tilde{h}_{1,q}^m + \tilde{h}_{1,s}^n)\pi i} \langle o^*(u^1)(v^-)^*, w^+ \rangle \\
= e^{-(\tilde{h}_{1,q}^m + \tilde{h}_{1,s}^n)\pi i} \mu (-1)^{[m(m+1)+n(n+1)]/4} (-1)^{\phi^+ o(a_{wta-1+wta^0-wta^1}u^0)w^+, w^+} \\
= e^{-(\tilde{h}_{1,q}^m + \tilde{h}_{1,s}^n)\pi i} \mu (-1)^{[m(m+1)+n(n+1)]/4+1} \beta ((v^+)^*, w^+).
\]

Therefore, we have

\[
\alpha = e^{-(h_{1,q}^m + h_{1,s}^n)\pi i} \mu \alpha, \quad \beta = e^{-(\tilde{h}_{1,q}^m + \tilde{h}_{1,s}^n)\pi i} \mu (-1)^{[m(m+1)+n(n+1)]/4+1} \beta
\]

and hence by (1.7)

\[
1 = e^{[(h_{1,q}^m + h_{1,s}^n) - (\tilde{h}_{1,q}^m + \tilde{h}_{1,s}^n)]\pi i} \cdot (-1)^{[m(m+1)+n(n+1)]/4+1} \\
= (-1)^{\gamma_{1,q,1,s} \cdot (-1)^{[m(m+1)+n(n+1)]/4+1} \\
= (-1)^{[(m-1)(q-1)+(n-1)(s-1)]/2+(q+s)/2}.
\]

Since $m \equiv 1 \mod 4$ and $n \equiv 1 \mod 4$, it is impossible if $(q+s)/2$ is odd. Thus we have

\[
I_U \left( \begin{array}{c} M_{1,1}^\varepsilon \\ W_{(1,q),(1,s)} M_{1,1}^\varepsilon \end{array} \right) = 0
\]

and hence

\[
I_U \left( \begin{array}{c} M_{1,1}^\varepsilon \\ W_{(1,q),(1,s)} M_{1,1}^\varepsilon \end{array} \right) \neq 0,
\]

if $(q+s)/2$ is odd.

\[\square\]
Finally, by using \( W_{(p,q),(r,s)} = W_{(1,q),(1,s)} \times W_{(p,1),(r,1)} \), \((M_{p,r})^\pm = W(p,r) \times (M_{1,1})^\pm\) for \(m,n \equiv 1 \) or \(2 \mod 4\), the \(S_3\)-symmetry of fusion rules [FHL, Li] and the associativity and commutativity of fusion product [H2], we can determine the fusion rules among all irreducible modules.

**Theorem 1.22.** For \(m,n \equiv 1\) or \(2 \mod 4\), \(1 \leq p < (m+2)/2\), \(1 \leq q < (m+3)/2\), \(1 \leq r < (n+2)/2\), \(1 \leq s < (n+3)/2\) such that
\[
\begin{align*}
q \text{ and } s \text{ are odd} & \quad \text{if } m \equiv 1 \text{ mod } 4 \text{ and } n \equiv 1 \text{ mod } 4 \\
q \text{ and } r \text{ are odd} & \quad \text{if } m \equiv 1 \text{ mod } 4 \text{ and } n \equiv 2 \text{ mod } 4 \\
p \text{ and } r \text{ are odd} & \quad \text{if } m \equiv 2 \text{ mod } 4 \text{ and } n \equiv 2 \text{ mod } 4,
\end{align*}
\]
we have the following fusion rules.

1. For any modules of nonsplit type,
\[
\dim I_U \begin{pmatrix}
W_{(p,q),(r,s)}
& W_{(p',q'),(r',s')}
& W_{(p'',q''),(r'',s'')}
\end{pmatrix}
= \begin{cases}
0 & \text{if neither } A \text{ nor } \tilde{A} \text{ is admissible}, \\
1 & \text{if either } A \text{ or } \tilde{A} \text{ is admissible but not both}, \\
2 & \text{if both } A \text{ and } \tilde{A} \text{ are admissible},
\end{cases}
\]
where \(A = \left( \left( (p,q), (p',q'), (p'',q'') \right), \left( (r,s), (r',s'), (r'',s'') \right) \right)\) and \(\tilde{A} = \left( \left( (\tilde{p},q), (p',q'), (p'',q'') \right), \left( (\tilde{r},s), (r',s'), (r'',s'') \right) \right)\).

2. For any modules \(W_{(p',q'),(r',s')}, W_{(p'',q''),(r'',s'')}\) of nonsplit type and \((M_{p,r})^\pm\) of split type,
\[
\dim I_U \begin{pmatrix}
M_{p,r}^+ & M_{p,r}^- \\
W_{(p',q'),(r',s')} & W_{(p'',q''),(r'',s'')}
\end{pmatrix}
= \dim I_U \begin{pmatrix}
M_{p,r}^+ & M_{p,r}^- \\
W_{(p'',q''),(r'',s'')} & W_{(p',q'),(r',s')}
\end{pmatrix}
= \dim I_U \begin{pmatrix}
W_{(p',q'),(r',s')} & W_{(p'',q''),(r'',s'')}
\end{pmatrix}
= 1.
if and only if both \( ((p, m), (p', q'), (p'', q'')) \) and \( ((r, n), (r', s'), (r'', s'')) \) satisfy the admissible condition.

3. For any modules \( W_{(p', q'), (r', s')} \) of nonsplit type and \((M_{p,r})^{\epsilon_2}, (M_{p', r'})^{\epsilon_1}\) of split type,

\[
\dim I_U \begin{pmatrix} (M_{p,r})^{\epsilon_2} \\ W_{(p', q'), (r', s')} & (M_{p', r'})^{\epsilon_1} \end{pmatrix} = \dim I_U \begin{pmatrix} (M_{p,r})^{\epsilon_2} \\ (M_{p', r'})^{\epsilon_1} W_{(p', q'), (r', s')} \end{pmatrix} = \dim I_U \begin{pmatrix} W_{(p', q'), (r', s')} \\ (M_{p', r'})^{\epsilon_1} ((M_{p,r})^{\epsilon_2})^* \end{pmatrix} = 1
\]

if and only if both \( ((p, m), (p', m), (p'', m)) \) and \( (r, n), (r', n), (r'', n) \) satisfy the admissible condition and

\[
\begin{cases} (-1)^{(p'+r')/2} = -\epsilon_1\epsilon_2, & \text{if } m \equiv 1 \mod 4 \text{ and } n \equiv 1 \mod 4, \\ (-1)^{(p'-r')/2} = \epsilon_1\epsilon_2, & \text{if } m \equiv 1 \mod 4 \text{ and } n \equiv 2 \mod 4, \\ (-1)^{(p'+r')/2} = -\epsilon_1\epsilon_2, & \text{if } m \equiv 2 \mod 4 \text{ and } n \equiv 2 \mod 4. 
\end{cases}
\]

4. For any modules of split type,

\[
\dim I_U \begin{pmatrix} ((M_{p', r'})^*)^* \\ (M_{p,r})^{\epsilon_1} (M_{p', r'})^{\epsilon_2} \end{pmatrix} = 1
\]

if and only if both \( ((p, m), (p', m), (p'', m)) \) and \( (r, n), (r', n), (r'', n) \) satisfy the admissible condition and \( \epsilon = \epsilon_1 = \epsilon_2 \).

For all other cases, \( I_U \left( \begin{array}{c} M_3 \\ M_1 \\ M_2 \end{array} \right) = 0 \).
Finally we will compute an example explicitly.

Let $m = 2$ and $n = 2$. In this case, we have $c_m = c_n = \frac{7}{10}$ and $U = L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right)$. Note that $L\left(\frac{7}{10}, 0\right)$ has exactly 6 irreducible modules and the highest weight $h_{r,s}$ is given by

\[
\begin{array}{c|c|c|c}
  s & r & 1 & 2 \\
  \hline
  1 & 0 & \frac{7}{10} & \frac{3}{2} \\
  2 & \frac{3}{5} & \frac{2}{80} & \frac{1}{10} \\
\end{array}
\]

**Theorem 1.23.** $U$ has 8 irreducible modules of nonsplit type, namely,

\[
\begin{align*}
W_{1,1,1,1} &= L(0\mid 0) \oplus L\left(\frac{3}{2} \mid \frac{3}{2}\right), \\
W_{3,1,1,1} &= L(0\mid \frac{3}{2}) \oplus L\left(\frac{3}{2} \mid 0\right), \\
W_{1,1,1,2} &= L\left(0 \mid \frac{3}{5}\right) \oplus L\left(\frac{3}{2} \mid \frac{1}{10}\right), \\
W_{3,1,1,2} &= L\left(0 \mid \frac{1}{10}\right) \oplus L\left(\frac{3}{2} \mid \frac{3}{5}\right), \\
W_{1,2,1,1} &= L\left(\frac{3}{5} \mid 0\right) \oplus L\left(\frac{1}{10} \mid \frac{3}{2}\right), \\
W_{3,2,1,1} &= L\left(\frac{1}{10} \mid 0\right) \oplus L\left(\frac{3}{5} \mid \frac{3}{2}\right), \\
W_{1,2,1,2} &= L\left(\frac{3}{5} \mid \frac{3}{5}\right) \oplus L\left(\frac{1}{10} \mid \frac{1}{10}\right), \\
W_{3,2,1,2} &= L\left(\frac{3}{5} \mid \frac{1}{10}\right) \oplus L\left(\frac{1}{10} \mid \frac{3}{5}\right).
\end{align*}
\]

Note that $p$ and $r$ must be both odd in this case.

**Theorem 1.24.** $U$ has 8 irreducible modules of twisted type, namely,

\[
\begin{align*}
M_{1,1}^\pm &= L\left(\frac{7}{16} \mid \frac{7}{16}\right)^\pm, \\
M_{1,2}^\pm &= L\left(\frac{7}{16} \mid \frac{3}{80}\right)^\pm, \\
M_{2,1}^\pm &= L\left(\frac{3}{80} \mid \frac{7}{16}\right)^\pm, \\
M_{2,2}^\pm &= L\left(\frac{3}{80} \mid \frac{3}{80}\right)^\pm.
\end{align*}
\]

**Remark 1.25.** By Theorem 1.22, it is easy to see that the VOA $U$ has a $\mathbb{Z}_4$ symmetry as follows:

\[
\begin{align*}
1 & \quad \text{on} \quad W_{1,1,1,1}, W_{1,2,1,1}, W_{1,1,1,2} \text{ and } W_{1,2,1,2}, \\
\sqrt{-1} & \quad \text{on} \quad M_{1,1}^+, M_{1,2}^+, M_{2,1}^+ \text{ and } M_{2,2}^+ \\
-1 & \quad \text{on} \quad W_{3,1,1,1}, W_{3,2,1,1}, W_{3,1,1,2} \text{ and } W_{3,2,1,2}, \\
-\sqrt{-1} & \quad \text{on} \quad M_{1,1}^-, M_{1,2}^-, M_{2,1}^- \text{ and } M_{2,2}^-.
\end{align*}
\]

It was shown in [LYY2] that the above symmetry is related to the $4B$-element of the Monster simple group.
2 Coxeter elements and McKay’s $E_8$-observation

Let $\mathbb{M}$ be the Monster simple group. It is known that the products of any two 2A-involutions of Monster simple group $\mathbb{M}$ fall into one of the following nine conjugacy classes [C, Atlas]:

$$1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, \text{ or } 3C.$$ 

Here, the first number denotes the order of elements in the conjugacy class and the second letter denotes the alphabetical order of the size of centralizer of the elements.

John McKay noticed that the orders of the elements in these conjugacy classes coincide with the multiplicities of the simple roots in a primitive isotropic element in an extended $E_8$-diagram $\hat{E}_8$ and he observed there is an interesting correspondence between these nine conjugacy classes of $\mathbb{M}$ with the nine nodes of the extended diagram $\hat{E}_8$ as follows:

\begin{equation}
\begin{array}{cccccccc}
1A & 2A & 3A & 4A & 5A & 6A & 4B & 2B \\
\end{array}
\end{equation}

$$3C$$

The above correspondence is often referred as to McKay’s $E_8$-observation. It suggests some relation between 2A-involutions of the Monster simple group and the root lattice of type $E_8$. A natural question is why the numerical labels of the extended $E_8$ diagram match exactly to the orders of the product of two 2A-involutions.

Our main aim is to give an explanation for this phenomenon via the theory of vertex operator algebras (VOA). It is well-known (cf. [FLM] and [Mi1]) that the Monster simple group $\mathbb{M}$ can be realized as the full automorphism group of the Moonshine VOA $V^{\natural} = \oplus_{n=0}^{\infty} V_{n}^{\natural}$. Moreover, for each 2A involution $\phi \in \mathbb{M}$, it is known that $\phi$ uniquely defines a special element $\epsilon = \epsilon(\phi) \in V_{2}^{\natural}$, which generates a subalgebra isomorphic to the simple Virasoro VOA $L(1/2, 0)$ of central charge $1/2$. The element $\epsilon$ is called a (rational) conformal vector of central charge $1/2$. In this case, the involution $\phi$ can be realized as the $\tau$-involution of $\epsilon$, i.e.,

$$\phi = \tau_{\epsilon} = \exp(16\pi i \epsilon(1)),$$

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where $\epsilon_{(1)}$ denotes the 1st operator of $\epsilon$ (cf. [C] and [Mi1]). In fact, there is a one to one correspondence between $2A$-involutions of the Monster and the conformal vectors of central charge $1/2$ in $V^\natural$ [Mi1]. Therefore, the main problem is to relate the extended $E_8$ diagram and its numerical labels to conformal vectors of central charge $1/2$ in $V^\natural$.

The key observation is that the numerical labels of the extended $E_8$ diagram correspond to the indexes of certain sublattices of the root lattice of type $E_8$. The coset structure of such a sublattice in $E_8$ induces a natural automorphism $\rho$ of the lattice VOA $V_{\sqrt{2}E_8}$. Moreover, $\rho$ is a product of two involutions. In other words, the numerical labels correspond exactly to the order of certain automorphisms of VOAs. As our main result, we will show that for a given node of the McKay diagram (1) labelled by $pX = 1A, 2A, \ldots, 2B$ or $3C$, the automorphism $\rho = \rho_{pX}$ naturally defines two conformal vectors $e$ and $f \in V^\natural$ of central charge $1/2$ and the product of their corresponding $\tau$-involutions is of the class $pX$.

Our method essentially follows the approach of [LYY1] and [LYY2] in which certain conformal vectors of central charge $1/2$ of the lattice VOA $V_{\sqrt{2}E_8}$ were constructed and studied. First, let us explain their results.

As long as there is no confusion, we will use the same notation to denote a root system, its Dynkin diagram and the corresponding root lattice. For example, $E_8$ denotes a root system, a Dynkin diagram, as well as its root lattice of type $E_8$. For a lattice $P$, we will use $\sqrt{2}P$ to denote the lattice which is isomorphic to $P$ as a $\mathbb{Z}$-module but with an inner product multiplied by 2.

Let $pX$ be the label of a node in McKay’s diagram $\hat{E}_8$. Let $\hat{E}_8 - \{pX\}$ be the sub Dynkin diagram of $\hat{E}_8$ obtained by removing the node $pX$. We will also use $\alpha_{pX}$ to denote the root associated with the node $pX$. In each case, $\hat{E}_8 - \{pX\}$ will be a Dynkin diagram associated with a semisimple root system and it generates a root sublattice $H_{pX}$ of index $p$ in $E_8$. For example, for $pX = 5A$, $H_{5A}$ is a root lattice of type $A_4 + A_4$.

The factor group $E_8/H_{pX}$ is a cyclic group of order $p$ and it induces a linear map

$$\varphi : \sqrt{2}E_8 \to \sqrt{2}E_8/\sqrt{2}H_{pX} \cong \mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p - 1\}.$$  \hfill (1.1)

For a lattice VOA $V_{\sqrt{2}E_8}$, the map $\rho_{pX}$ defined by

$$\rho_{pX}(u \otimes e^\alpha) = e^{2\pi \sqrt{-1} \varphi(\alpha)/p} u \otimes e^\alpha \quad \text{for} \quad \alpha \in \sqrt{2}E_8 \text{ and } u \in M(1)^{\otimes 8}$$  \hfill (1.2)
is an automorphism of order $p$. (See §2 for the definition of lattice VOAs.)

By a result in [DLMN], it is known that the lattice VOA $V_{\sqrt{\mathbb{E}_8}}$ contains a special conformal element $\epsilon_{\sqrt{\mathbb{E}_8}}$ which is fixed by the Weyl group. In this case, the $\tau$-involution $\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}}$ acts as $-1$ on the weight one space $(V_{\sqrt{\mathbb{E}_8}})_1$ and $\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}}$ inverts $\rho_{pX}$ for each node $pX$, i.e.,

$$\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \rho_{pX} \tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} = \rho_{pX}^{-1}. \quad (1.3)$$

Thus, the group $<\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}}, \rho_{pX}>$ generated by $\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}}$ and $\rho_{pX}$ is isomorphic to the dihedral group of order $2p$ and $\rho_{pX}$ is a product of 2 involutions.

Now let $f_{\sqrt{\mathbb{E}_8}} = \rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}})$. Then, we have

$$\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}}) \tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} = \tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \left(\rho_{pX} \tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \rho_{pX}^{-1}\right) \rho_{pX}^{-1} = \rho_{pX}^{-2}. \quad (1.3)$$

Hence, if $p$ is an odd integer, then $\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}})$ is of order $p$, but if $p$ is even, then the product is of order $p/2$ only. However, it is natural since $z = (\rho_{pX})^{p/2}$ is inside the center of $<\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}}, \tau_{f_{\sqrt{\mathbb{E}_8}}}>$ and $\epsilon_{\sqrt{\mathbb{E}_8}}, f_{\sqrt{\mathbb{E}_8}} \in (V_{\sqrt{\mathbb{E}_8}})^{<>}$.

In [LYY1] and [LYY2], the vertex subalgebra and the Griess subalgebra generated by $\epsilon_{\sqrt{\mathbb{E}_8}}$ and $f_{\sqrt{\mathbb{E}_8}}$ has been studied very carefully. It was shown that the relation between the conformal vectors $\epsilon_{\sqrt{\mathbb{E}_8}}$ and $f_{\sqrt{\mathbb{E}_8}}$ in the weight two space of $(V_{\sqrt{\mathbb{E}_8}})_2$ agrees with the calculation of Conway [C] on the Monstrous Griess algebra. Namely, they showed that the Griess algebra generated the pairs of conformal vectors $\epsilon_{\sqrt{\mathbb{E}_8}}$ and $f_{\sqrt{\mathbb{E}_8}}$ in $V_{\sqrt{\mathbb{E}_8}}$ is isomorphic to the Griess algebra generated by two conformal vectors $\epsilon'$, $f'$ in $V^2$ such that $\tau_{\epsilon' \tau f'}$ is of class $pX$.

Based on these facts, it was conjectured in [LYY1, LYY2] that the vertex subalgebra $VA(\epsilon_{\sqrt{\mathbb{E}_8}}, f_{\sqrt{\mathbb{E}_8}})$ generated by $\epsilon_{\sqrt{\mathbb{E}_8}}, \rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}})$ is isomorphic to $VA(\epsilon', f')$ as VOAs.

We proved this conjecture.

**Main theorem:** Let $\epsilon'$ and $f'$ be two conformal vectors in the Moonshine VOA $V^2$ such that $\tau_{\epsilon'} \tau f'$ is of class $pX$. Then the subVOA $VA(\epsilon', f')$ generated by $\epsilon'$ and $f'$ in $V^2$ is isomorphic to the subVOA $VA(\epsilon_{\sqrt{\mathbb{E}_8}}, f_{\sqrt{\mathbb{E}_8}})$ generated by $\epsilon_{\sqrt{\mathbb{E}_8}}, \rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}})$ in $V_{\sqrt{\mathbb{E}_8}}$. As a consequence, the product of the $\tau$-involutions $\tau_{\epsilon_{\sqrt{\mathbb{E}_8}}} \tau_{\rho_{pX}(\epsilon_{\sqrt{\mathbb{E}_8}})}$ induces an automorphism of class $pX$ in $\text{Aut } V^2$. 

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Remark 1. By the construction of Frenkel-Lepowsky-Meurman [FLM], the Moonshine VOA $V^\natural$ can be written as

$$V^\natural = V^+_\Lambda \oplus (V^T_\Lambda)^+,$$

where $\Lambda$ is the Leech lattice, $V^+_\Lambda$ is the fixed point subalgebra of the Leech lattice VOA $V_\Lambda$ by an automorphism $\theta$ induced by an isometry $\beta \mapsto -\beta$ for $\beta \in \Lambda$, and $(V^T_\Lambda)^+$ is an irreducible module of $V^+_\Lambda$.

Set $W = VA(\epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8}))$. Since $\sqrt{2}E_8 \subseteq \Lambda$, it is clear that $W \subseteq V_{\sqrt{2}E_8} \subseteq V_\Lambda$.

Nevertheless, $\rho_{pX}$ does not commute with $\theta$ and thus $f_{\sqrt{2}E_8} = \rho_{pX}(\epsilon_{\sqrt{2}E_8})$ is, in general, not contained in $V_{\sqrt{2}E_8}$ nor $V^+_\Lambda$. On the other hand, it is not difficult to show that there is an automorphism $\xi$ of order $p$ of $V_{\sqrt{2}E_8}$ (and of $V_\Lambda$) such that the algebra

$$W \subseteq V^{<\xi>}_{\sqrt{2}E_8} \subseteq V^{<\xi>}_\Lambda.$$

Therefore, if we can construct $V^\natural$ from $V_\Lambda$ by $\xi$-orbifold construction, then we have $W \subseteq V^\natural$ naturally. The problem is that we don’t know if such orbifold constructions exist except for a $2B$-orbifold construction.

By the properties of the Conway group and the character table in [Atlas], it is also not hard to find two conformal elements $\epsilon, f_{pX}$ in $V^+_\Lambda$ such that $\tau_\epsilon \tau_{f_{pX}}$ belongs to the class $pX$ in $Aut(V^\natural) \cong M$. However, there is a crucial difference between these two pairs of conformal elements $\{\epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8})\} \subseteq V_{\sqrt{2}E_8} \subseteq V_\Lambda$ and $\{\epsilon, f_{pX}\} \subseteq V^+_\Lambda \subseteq V_\Lambda$. First we note that $\epsilon_{\sqrt{2}E_8}$ and $\rho_{pX}(\epsilon_{\sqrt{2}E_8})$ acts as $-1$ on the weight one space $(V_{\sqrt{2}E_8})_1$ (8-dimensional) and so they have the same 8-dimensional eigenspace in $(V_\Lambda)_1$ with eigenvalue $-1$. On the other hand, $\epsilon$ and $f_{pX}$ in $(V_\Lambda)^+_1$ have different eigenspaces (if $pX \neq 1A$) in $(V_\Lambda)_1$ with eigenvalue $-1$. Furthermore, $\tau_{\epsilon_{\sqrt{2}E_8}} \tau_{\rho_{pX}(\epsilon_{\sqrt{2}E_8})}$ is a diagonal automorphism, that is, it is given as $\exp(v_{(0)})$ by $v \in CE_8$, but $\tau_{\epsilon} \tau_{f_{pX}}$ is defined by an automorphism of the Leech lattice $\Lambda$. Therefore, our strategy is to transform $\epsilon_{\sqrt{2}E_8}$ and $\rho_{pX}(\epsilon_{\sqrt{2}E_8})$ so that they have different eigenspaces with eigenvalue $-1$ and $\rho_{pX}$ can be viewed as an automorphism of some lattice.

The above facts suggest the existence of orbifold construction for a quite large part of $V_\Lambda$ into $V^\natural$. We will prove our main theorem from this point of view.
It follows from the definition of \( \rho_{pX} \) that \( VA(\epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8})) \) is isomorphic to a subVOA of \( VA(\epsilon_{\sqrt{2}E_8}, \rho_{6A}(\epsilon_{\sqrt{2}E_8})) \) for \( pX = 1A, 2B, 3A \) and \( VA(\epsilon_{\sqrt{2}E_8}, \rho_{2A}(\epsilon_{\sqrt{2}E_8})) \) is isomorphic to a subVOA of \( VA(\epsilon_{\sqrt{2}E_8}, \rho_{4B}(\epsilon_{\sqrt{2}E_8})) \) since 

\[ E_8 \subseteq A_2 + A_5 + A_1 \subseteq A_2 + E_6, A_3 + D_5 \subseteq D_8, \text{ and } A_1 + A_7 \subseteq A_1 + E_7. \]

Therefore, we will prove only the cases for \( pX = 4A, 5A, 6A, 4B, 3C \).

### 2.1 Lattice VOA \( V_L \) and conformal vectors

#### 2.1.1 Lattice VOA

First, we will briefly recall the construction of a lattice VOA \( V_L \) associated with a rank \( n \) positive definite even lattice \( L \) with an inner product \( \langle \cdot, \cdot \rangle \) from [FLM].

Consider \( H = \mathbb{C} \otimes \mathbb{Z} L \) as an abelian Lie algebra and extend the inner product \( \langle \cdot, \cdot \rangle \) \( \mathbb{C} \)-linearly. Let \( \hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \) be its affine Lie algebra with the Lie bracket

\[
[a \otimes t^n, b \otimes t^m] = \delta_{n+m,0} n \langle a, b \rangle.
\]

Then

\[
M(1)^{\otimes n} = \mathbb{C}[\alpha(n) \mid \alpha \in H, n < 0] \cdot 1
\]

is the unique irreducible \( \hat{H} \)-module such that \( \alpha(n)1 = 0 \) for \( \alpha \in H, n \geq 0 \), where \( \alpha(n) = \alpha \otimes t^n \). (Although the original notation is \( M(1) \) in [FLM], we will use \( M(1)^{\otimes n} \) in order to express the rank).

Let \( \mathbb{C}\{L\} = \text{span}_\mathbb{C}\{e^\alpha \mid \alpha \in L\} \) denotes a twisted group algebra of \( L \) such that

\[
e^{\alpha}e^{\beta} \in \mathbb{Z}e^{\alpha+\beta} \quad \text{and} \quad e^{\alpha}e^{\beta} = (-1)^{\langle \alpha, \beta \rangle}e^{\beta}e^{\alpha}.
\]

Usually, we first consider a central extension

\[
1 \rightarrow < \kappa > \rightarrow \hat{L} \rightarrow L \rightarrow 1 \quad \text{with } \kappa^2 = 1.
\]

Let \( \mathbb{C} \) be a one dimensional module of \( < \kappa > \) such that \( \kappa \) acts on \( \mathbb{C} \) as \(-1\). Then \( \mathbb{C}\{L\} \) is defined to be the induced module

\[
\mathbb{C}\{L\} = \text{Ind}_{< \kappa >}^{\hat{L}} \mathbb{C} = \mathbb{C}[\hat{L}] / < \kappa + 1 > \quad (\cong \text{span}_\mathbb{C}\{e^\alpha \mid \alpha \in L\} \text{ as a vector space}).
\]
Then
\[ V_L = \bigoplus_{\alpha \in L} M(1)^{\otimes n} \otimes e^\alpha \]
has a VOA structure, that is, for \( v, u \in V_L \) and any integer \( n \in \mathbb{Z} \), \( V_L \) has a bilinear product \( v_{(n)}u \in V_L \) satisfying several conditions. We simply denote \( 1 \otimes e^\alpha \) by \( e^\alpha \) and \( 1 \otimes e^0 \) by \( 1 \).

Viewing \( v_{(n)} \in \text{End}(V) \), the generating function
\[ Y(v, z) = \sum v_{(n)} z^{-n-1} \in \text{End}(V)[[z, z^{-1}]] \]
is called the vertex operator of \( v \).

The VOA \( V_L \) has a natural \( \mathbb{N} \)-grading such that
\[ V_L = \bigoplus_{n=0}^{\infty} (V_L)_n, \]
where the weight of an element is defined by
\[ \text{wt}(\alpha_1(-n_1) \cdots \alpha_r(-n_r)e^\beta) = n_1 + \cdots + n_r + \frac{\langle \beta, \beta \rangle}{2}. \]
It is easy to show that \( (V_L)_0 = C1 \) and \( (V_L)_1 = \sum_{\alpha \in L} C_\alpha (1-1)1 + \sum_{\alpha \in \Phi(L)} C_e^\alpha \), where \( \Phi(L) \) denotes the set of all roots. The vertex operators for a weight one element \( a(-1)1 \) and \( e^\alpha \) (\( \alpha \) a root) are defined by
\[ Y(a(-1)1, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \]
\[ Y(e^\alpha, z) = \exp \left( \sum_{n \in \mathbb{Z}^+} \frac{a(-n)}{n} z^n \right) \exp \left( \sum_{n \in \mathbb{Z}^+} \frac{a(n)}{-n} z^{-n} \right) e^\alpha z^\alpha, \]
see [FLM] for the detail.

For any VOA \( V = \bigoplus_{n=0}^{\infty} V_n \) with \( \dim V_0 = 1 \), it is well known [FLM] that the weight one space \( V_1 \) is a Lie algebra with the bracket \( [u, v] = u_{(0)} v \) and with an invariant bilinear form given by \( (v, u)1 = v_{(1)} u \). In particular, if \( L \) is a root lattice of type \( A_n, D_n \) or \( E_n \), then \( (V_L)_1 \) is a simple Lie algebra, where \([M(1)^{\otimes n}]_1 \) is a Cartan subalgebra and \( \{e^\alpha | \alpha \in L, \langle \alpha, \alpha \rangle = 2 \} \) is the set of root vectors.

For any VOA \( V \) and \( v \in V_1 \), since \( [v_{(0)}, u_{(m)}] = (v_{(0)} u)_{(m)}, \)
\[ \exp(v_{(0)})(u_{(m)} w) = (\exp(v_{(0)} u)_{(m)})(\exp(v_{(0)} w) \quad \text{for any } u, w \in V. \]
Furthermore, since \( v(0)\omega = -\omega(0)v + L(-1)\omega(1)v - \frac{L(-1)^2}{2}\omega(2)v + ... = 0 \) for a Virasoro element \( \omega \) by the skew-symmetry, we have \( \exp(v(0))\omega = \omega \). Therefore, \( \exp(v(0)) \) is always an automorphism of \( V_L \) for any \( v \in (V_L)_1 \). As an application, we can induce any inner automorphism of Lie algebra \( V_1 \) to an automorphism of \( V \).

Now let \( \bar{\cdot} : \hat{L} \to L \) be the natural projection of \( \hat{L} \) to \( L \) and let \( \iota : a \in L \to e^a \in \hat{L} \) be a section, i.e. \( \bar{\iota} \) is \( \text{id}_L \). We denote \( \Im \iota \) by \( e^L \). For any \( g \in \text{Aut}\hat{L} \), define \( \bar{g} = \bar{\iota} \circ g \circ \iota \in \text{Aut}L \).

The following lemma can be proved easily from the construction (cf. [FLM]).

**Lemma 1.** For any \( \mu \in \text{Aut}(\hat{L}, \langle , \rangle) = \{ g \in \text{Aut} \hat{L} | \langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle \text{ for all } \alpha, \beta \in L \} \), we can define an automorphism \( \bar{\mu} \) of \( V_L \) naturally by

\[
\bar{\mu}(\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^a) = (\bar{\mu}\alpha_1)(-n_1) \cdots (\bar{\mu}\alpha_k)(-n_k) \otimes \mu(e^a),
\]

where \( \alpha_1, \ldots, \alpha_k \in L \) and \( e^a \in \hat{L} \). On the other hand, if \( \tau \in \text{Aut}(V_L) \) keeps \( (M(1)_{\otimes n})_1 \) invariant, then there are \( \mu \in \text{Aut}(\hat{L}) \) and \( b \in (M(1)_{\otimes n})_1 = CL \) such that \( \tau = \bar{\mu} \cdot \exp(b(0)) \).

From now on, we won’t distinguish \( \bar{\mu} \) and \( \mu \) and denote both of them by \( \mu \). As an important example of \( \text{Aut}(V_L) \), if a lattice \( L \) is an even lattice, then we are able to extend the \(-1\) map on \( L \) to an automorphism \( \theta \) of \( V_L \) by

\[
\theta : \alpha_1(i_1) \cdots \alpha_k(i_k)e^a \to (-1)^{k+(a,a)/2}\alpha_1(i_1) \cdots \alpha_k(i_k)e^{-a} \quad \text{for } \alpha_1, ..., \alpha_k, \alpha \in L,
\]

(see [FLM]).

If \( L = A_n \) or \( D_n \), then by viewing the weight one space \( (V_L)_1 \) as a Lie algebra \( G = sl_{n+1}(\mathbb{C}) \) (or \( o_{2n}(\mathbb{C}) \)), the restriction of \( \theta \) on \( (V_L)_1 \) coincides with a Cartan involution:

\[
\theta : A \to -^tA \quad \text{for } A \in G
\]

after suitable rearrangement of roots.

### 2.1.2 Conformal element with central charge 1/2

For \( v \in L \) with \( \langle v, v \rangle = 4 \),

\[
\epsilon_v^\pm = \frac{1}{16}v(-1)v(-1)1 \pm \frac{1}{4}(e^v + e^{-v}) \in (V_L)_2
\]

are rational conformal elements with central charge 1/2, that is, \( VA(\epsilon_v^\pm) \) are isomorphic to a simple Virasoro VOA \( L(\frac{1}{2}, 0) \), which describes the critical points of the 2-dimensional
Ising model (cf. [DMZ, Mi1]). If \( L = \sqrt{2}R \) and \( R \) is a root lattice, then Dong, Li, Mason and Norton show that \( VA(< \epsilon_{\sqrt{2}v} | v \in \Phi(R) >) \) is a subVOA of \( V_L \) with a Virasoro element \( \omega^- \) and \( \omega - \omega^- \) is also a conformal vector. When \( L = \sqrt{2}E_8 \), \( e = \omega - \omega^- \) is a rational conformal element with central charge \( \frac{1}{2} \). By the definition, we have

\[
\epsilon_{\sqrt{2}E_8} = \frac{1}{16} \omega_{\sqrt{2}E_8} + \frac{1}{32} \sum_{\alpha \in \Phi(E_8)^+} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}),
\]

where \( \omega_{\sqrt{2}E_8} = \frac{1}{60} \sum_{\alpha \in \Phi(E_8)^+} \alpha(-1)^2 \mathbf{1} \) is the Virasoro of the lattice VOA \( V_{\sqrt{2}E_8} \).

From the definition of \( \epsilon_{\sqrt{2}E_8} \), \( \epsilon_{\sqrt{2}E_8} \) is orthogonal to all \( \epsilon^{-\sqrt{2}\alpha}, \alpha \in \Phi(E_8) \), which means,

\[
(\epsilon_{\sqrt{2}E_8})^{(n)} \epsilon^{-\sqrt{2}\alpha} = 0 \quad \text{for all } n \geq 0
\]

On the other hand, from the definition \( \rho_{pX} \), \( \rho_{pX} \) fixes all \( \epsilon^{-\sqrt{2}\alpha}, \alpha \in \Phi(H_pX) \), and hence \( \rho_{pX}(\epsilon_{\sqrt{2}E_8}) \) is also orthogonal to all \( \epsilon^{-\sqrt{2}\alpha} (\alpha \in \Phi(H_pX)) \). In other words,

\[
VA(\epsilon_{\sqrt{2}E_8}; \rho_{pX}(\epsilon_{\sqrt{2}E_8})) \otimes VA(\epsilon^{-\sqrt{2}\alpha} : \alpha \in \Phi(H_pX)) \subseteq V_{\sqrt{2}E_8},
\]

where VA(S) denotes a vertex subalgebra generated by S.

We first note that we can naturally induce an automorphism of \( \sqrt{2}L \) from one of \( L \). Furthermore, since \( \sqrt{2}L \) is abelian, we can induce all automorphisms of \( \sqrt{2}L \) to those of \( V_{\sqrt{2}L} \). Since the root lattice \( H_pX \) is of rank 8, a Coxeter element \( \xi_S = Rf_{x_1} \cdots Rf_{x_8} \) of a fundamental root system \( S = \{x_1, ..., x_8\} \) of \( H_pX \) acts on the weight one space \( (V_{\sqrt{2}E_8})_1 \) of \( V_{\sqrt{2}E_8} \) fixed point freely and fixes all elements in VA(\( \epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8}) \)). Therefore, VA(\( \epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8}) \)) \subseteq V_{\sqrt{2}E_8}^{|\xi_S|} \) has no weight one space and the weight two space \( VA(\epsilon_{\sqrt{2}E_8}, \rho_{pX}(\epsilon_{\sqrt{2}E_8}))_2 \) becomes a commutative algebra (Griess algebra) with a product \( \times \) defined by \( v \times u = v(1)u \) and an inner product \( \langle v, u \rangle_1 = v(3)u \).

### 2.2 Weight one space \((V_L)_1\)

In this section, we will study three important automorphisms \( \xi, \sigma, \theta \) of a Niemeier lattice VOA \( V_N \). The fundamental fact is that their behavior are almost determined by the actions on the weight one space \( (V_N)_1 \), which is a direct sum of simple Lie algebras.

We will consider only Niemeier lattices with the root systems \( A_4^6, A_8^3, A_5^4, D_4, A_7^2D_5^2 \).
Let \( v_1, v_2, \ldots, v_{n+1} \) denote an orthonormal basis and set
\[
a_i = v_i - v_{i+1} \quad (i = 1, \ldots, n) \\
d_1 = -v_1 - v_2, \ d_2 = v_1 - v_2, \ldots, \ d_n = v_{n-1} - v_n.
\]
Then
\[
a_1 \ a_2 \ \cdots \ a_{n-1} \ a_n
\]
is a fundamental root system of type \( A_n \) and
\[
d_1 \ d_2 \ d_3 \ \cdots \ d_{n-1} \ d_n
\]
is one of type \( D_n \).

If \( L \) is a root lattice of type \( A_n \) or \( D_n \), we will introduce formal elements \( e^{\pm v_i} \) such that
\[
e^{v_i} e^{-v_i} = e^{-v_i} e^{v_i} = 1, \quad e^{\pm v_i} e^{\pm v_j} = -e^{\pm v_j} e^{\pm v_i} \quad (\text{for } i \neq j).
\]
For \( \alpha = \lambda_1 v_1 + \cdots + \lambda_{n+1} v_{n+1} \in L \), we set \( e^\alpha = e^{\lambda_1 v_1} \cdots e^{\lambda_{n+1} v_n} \). Then \( \{ \pm e^\alpha \mid \alpha \in L \} \) is a central extension of \( L \) by \( \mathbb{Z}/2\mathbb{Z} \) such that \( e^\alpha e^\beta = (-1)^{\langle \alpha, \beta \rangle} e^\beta e^\alpha \). By abuse of notation, we will identify the symbols \( \pm e^\alpha \) with the elements in \( \hat{L} \subseteq \mathbb{C}\{L\} \). For convenience, we will also use \( e^{v_i+v_j} \) to denote \( e^{v_i} e^{v_j} \). Note that \( e^{v_i+v_j} = e^{v_j} e^{v_i} = -e^{v_i} e^{v_j} = -e^{v_i+v_j} \) if \( i \neq j \).

### 2.2.1 Coxeter elements on Lie algebra

Let \( L \) be a root lattice of type \( A_n, D_n \) or \( E_n \) and \( \{ \alpha_1, \ldots, \alpha_n \} \) a set of fundamental roots. Let \( R_{f\alpha_1}, R_{f\alpha_2}, \ldots, R_{f\alpha_n} \) be the simple reflections associated with the roots \( \alpha_1, \ldots, \alpha_n \), respectively, i.e.,
\[
R_{f\alpha} (\beta) = \beta - \langle \beta, \alpha \rangle \alpha \quad \text{for any root } \beta \in L.
\]
Then an automorphism
\[
\xi_R = R_{f\alpha_n} \cdots R_{f\alpha_2} R_{f\alpha_1}
\]
is called a “Coxeter element.”

**Remark 2.** Note that the definition of a Coxeter element depends on the choice as well as the order of the fundamental roots \( \{ \alpha_1, \ldots, \alpha_n \} \). However, it can be shown that all Coxeter elements are conjugate in the Weyl group. (cf. [H])
We will study only the cases for $A_n$ and $D_n$ ($n = 4, 5$). Although a Coxeter element $\xi_L$ is only an automorphism of a root lattice $L$, it can extended naturally an automorphism $\xi_L$ of the corresponding Lie algebra $\mathcal{G}$.

For an even lattice $N$ with a root lattice $L$ of full rank $n$, the weight one space $(\mathcal{V}_N)_1 = (\mathcal{V}_L)_1$ has a semisimple Lie algebra structure. Note that

$$(M(1)^{\otimes n})_1 = \{v(-1)1 \mid v \in \mathbb{C}N\}$$

is a Cartan subalgebra and $\{e^\alpha \mid \alpha \text{ a root}\}$ is the set of all root vectors. Moreover, the action of $\xi_L$ on $M(1)^{\otimes n}$ can be defined naturally by its action on $L$.

**The case for type $A_n$:**

Let $\mathcal{G}$ be a Lie algebra of type $A_n$ with a root lattice $L$. Then a Coxeter element $\overline{\xi_{A_n}} = Rf_{a_1} \cdots Rf_{a_n}$ is given by

$$\begin{cases}
a_i \rightarrow a_{i+1} & \text{for } i = 1, \ldots, n \\
a_n \rightarrow -a_1 - a_2 - \cdots - a_n
\end{cases}$$

By using the orthonormal basis $\{v_1, \ldots, v_n\}$, it is given by

$$\overline{\xi_{A_n}}(v_i) = v_{i+1} \quad \text{and} \quad \overline{\xi_{A_n}}(v_{n+1}) = v_1 \text{ for } i = 1, \ldots, n.$$ 

Therefore, the map $\xi_{A_n}$ defined by

$$\xi_{A_n}(e^{\pm v_i + v_{i+1}}) = e^{\pm v_{i+1} + v_{i+2}}, \xi_{A_n}(e^{v_n - v_{n+1}}) = -e^{-v_1 + v_{n+1}} \quad \text{and} \quad \xi_{A_n}(e^{-v_1 + v_{n+1}}) = e^{-v_2 + v_1}$$

is an automorphism of $\hat{L} = \{\pm e^\alpha \mid \alpha \in L\}$ (and also one of $V_L$).

As it is well known, a Lie algebra $\mathcal{G}_{A_n}$ of type $A_n$ is isomorphic to

$$\text{sl}(n+1, \mathbb{C}) = \{F \in M_{n+1,n+1}(\mathbb{C}) \mid \text{tr}F = 0\}$$

and the set $T$ of all diagonal matrices with trace 0 is a Cartan subalgebra. Under this identification, we have an isomorphism $\phi : \text{sl}(n+1, \mathbb{C}) \rightarrow (\mathcal{V}_L)_1$ given by $\phi(\overline{E_{ii}} - \overline{E_{jj}}) = (v_i(-1)1 - v_j(1)1)$ and $\phi(E_{ij}) = e^{v_i - v_j}$, where $E_{ij}$ denotes a matrix with 1 at $(i, j)$-entry and zero elsewhere. Note that $\phi(T) = (M(1)^{\otimes n})_1$. 

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Let $\omega = e^{2\pi \sqrt{-1}/(n+1)}$ and set

$$P = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 1 \\
1 & 0 & \cdots & 0
\end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix}
\omega & \omega^2 & \cdots & \omega^n & 1 \\
\omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\omega^n & \omega^{2n} & \cdots & \omega^{n^2} & 1 \\
1 & 1 & \cdots & 1 & 1
\end{pmatrix}.$$

Then the action of $\xi_{An}$ on $G$ is given by the conjugation of $P$, that is,

$$\xi_{An} : A \to P^{-1}AP \quad \text{for} \quad A \in sl(n+1, \mathbb{C})$$

and

$$B^{-1}PB = \text{diag}(\omega, \omega^2, \ldots, 1)$$

is a diagonal matrix. Define a map $\sigma_{An} : sl(n+1, \mathbb{C}) \to sl(n+1, \mathbb{C})$ by $\sigma_{An}(A) = B^{-1}AB$.

Since $\mathcal{C} = \langle P, P^2, \ldots, P^n \rangle = \mathcal{G}^{<\xi_{An}>}$ and $\sigma_{An}(\mathcal{C}) = T, \mathcal{C}$ is another Cartan subalgebra. It also follows from a direct calculation that

$$\sigma_{An}\xi_{An}\sigma_{An}^{-1}(E_{ij}) = B^{-1}P^{-1}BE_{st}B^{-1}PB = \omega^{j-i}E_{ij}.$$

We note that $B$ is a symmetric matrix and

$$tBB = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix} = (\delta_{i+j\equiv 0 \pmod{n+1}})_{i,j=1,\ldots,n+1}$$

is a permutation matrix of order 2. Moreover,

$$\theta\sigma_{An} \theta\sigma_{An}^{-1}(A) = -t(B^{-1}(-t(BAB^{-1}))B) = (tBB)A(tBB)^{-1} \quad \text{for} \quad A \in \mathcal{G}.$$

**Lemma 2.** $\theta$ and $\sigma_{An} \theta\sigma_{An}^{-1}$ commute with each other and we have

$$\begin{cases}
\sigma_{An} \theta\sigma_{An}^{-1}(E_{ij}) = -E_{n+1-j,n+1-i} \quad \text{for} \quad 1 \leq i, j \leq n, \\
\sigma_{An} \theta\sigma_{An}^{-1}(E_{i,n+1}) = -E_{n+1,n+1-i}, \\
\sigma_{An} \theta\sigma_{An}^{-1}(E_{n+1,j}) = -E_{n+1-j,n+1}, \\
\sigma_{An} \theta\sigma_{An}^{-1}(E_{n+1,n+1}) = -E_{n+1,n+1}.
\end{cases}$$
Thus by identifying $e^{a_i}$ with $E_{i,i+1}$, we also have

$$\sigma_A \theta \sigma_A^{-1} (e^{a_i}) = e^{a_{n-i}} \text{ for } i = 1, \ldots, n - 1,$$

$$\sigma_A \theta \sigma_A^{-1} (e^{a_n}) = -e^{-a_1 - a_2 - \cdots - a_n}$$

The case for type $D_4$:

In the case of $D_4$, the action of a Coxeter element $\xi_{D_4}$ is

$$\text{Rf}_d \text{Rf}_d \text{Rf}_d \text{Rf}_d : \begin{cases} 
  d_1 \to d_2 + d_3 \\
  d_2 \to d_1 + d_3 \\
  d_3 \to d_4 \\
  d_4 \to -d_1 - d_2 - d_3 - d_4.
\end{cases}$$

If we denote it by using an orthonormal basis $\{v_1, v_2, v_3, v_4\}$, the action is given by

$$\overline{\xi}_{D_4} : \begin{cases}
  v_1 \to -v_1, \\
  v_2 \to v_3 \to v_4 \to -v_2 \to -v_3 \to -v_4 \to v_2.
\end{cases}$$

It induces an action on the symbols $\{\pm e^{\pm v_i}\}$ as follows:

$$\xi_{D_4} : \begin{cases}
  e^{v_1} \to e^{-v_1} \to e^{v_1}, \\
  e^{v_2} \to e^{v_3} \to e^{v_4} \to e^{-v_2} \to e^{-v_3} \to e^{-v_4} \to e^{v_2}.
\end{cases}$$

Thus we can define an automorphism $\xi_{D_4}$ of $\hat{L}$ (and also of $V_L$) by

$$\xi_{D_4} : \begin{cases}
  e^{v_1+v_2} \to e^{-v_1+v_3} \to e^{v_1+v_4} \to e^{-v_1-v_2} \to e^{v_1-v_3} \to e^{-v_1-v_4} \to e^{v_1+v_2}, \\
  e^{-v_1+v_2} \to e^{v_1+v_3} \to e^{-v_1+v_4} \to e^{v_1-v_2} \to e^{v_1-v_3} \to e^{-v_1-v_4} \to e^{-v_1+v_2}, \\
  e^{v_2+v_3} \to e^{v_3+v_4} \to e^{v_4-v_2} = -e^{-v_2+v_3} \to e^{-v_2-v_3} \to e^{-v_3-v_4} \to e^{v_2-v_3}, \\
  e^{v_2-v_3} \to e^{v_3-v_4} \to -e^{v_2+v_3} \to e^{v_2-v_3} \to -e^{v_2-v_4} \to e^{v_2-v_3}, \\
  v_1(-1)1 \to -v_1(-1)1, \\
  v_2(-1)1 \to v_3(-1)1 \to v_4(-1)1 \to -v_2(-1)1.
\end{cases}$$

Note that the fixed point space of $\xi_{D_4}$ is spanned by

$$e^{v_1+v_2} + e^{-v_1+v_3} - e^{v_1+v_4} - e^{-v_1-v_2} + e^{v_1-v_3} + e^{-v_1-v_4},$$

$$e^{v_1-v_2} + e^{-v_1-v_3} - e^{v_1-v_4} + e^{-v_1+v_2} - e^{v_1+v_3} - e^{-v_1+v_4},$$

$$e^{v_2+v_3} + e^{v_1+v_4} + e^{v_4-v_2} + e^{-v_2-v_3} + e^{v_3-v_4} - e^{-v_4+v_2},$$

and

$$e^{v_2-v_3} + e^{v_3-v_4} - e^{v_4+v_2} - e^{-v_2+v_3} - e^{-v_3+v_4} - e^{-v_4-v_2}.$$
As it is well known, a Lie algebra of type $D_n$ is isomorphic to
\[
\mathfrak{o}_{2n}(\mathbb{C}) = \left\{ A = \begin{pmatrix} E & C \\ D & -tE \end{pmatrix} \middle| E, C, D \in M_n(\mathbb{C}), \ tC = -C, \ tD = -D \right\}.
\]

Using $w = e^{2\pi\sqrt{-1}/3}$, set

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \frac{1}{\sqrt{6}} \begin{pmatrix}
\sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & -1 & 1 \\
0 & 1 & w & w^2 & 0 & -1 & w & -w^2 \\
0 & 1 & w^2 & w & 0 & 1 & -w^2 & w \\
-\sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & -1 & 1 & -1 \\
0 & 1 & w & w^2 & 0 & 1 & -w & w^2 \\
0 & 1 & w^2 & w & 0 & -1 & w^2 & -w
\end{pmatrix}
\]

and define a map $\sigma_{D_4} : A \to B^{-1}AB$. Then the Coxeter element $\xi_{D_4}$ is given by $A \to P^{-1}AP$ and $\sigma_{D_4}(A)$ is a diagonal matrix for $A \in (\mathfrak{o}_8(\mathbb{C}))^{<\xi_{D_4}>}$, which implies that $(\mathfrak{o}_8(\mathbb{C}))^{<\xi_{D_4}>}$ is a Cartan subalgebra of $\mathfrak{o}_8(\mathbb{C})$. Note that

\[
{^tBB} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

is a permutation matrix of order 2 and so we have:

**Lemma 3.** $\theta$ and $\sigma_{D_4} \theta \sigma_{D_4}^{-1}$ commute with each other and

\[
\begin{align*}
\sigma_{D_4} \theta \sigma_{D_4}^{-1} & (e^{d_i}) = -e^{-d_i} \text{ for } i = 1, 2 \\
\sigma_{D_4} \theta \sigma_{D_4}^{-1} & (e^{d_3}) = -e^{d_1+d_2+d_3+d_4} \\
\sigma_{D_4} \theta \sigma_{D_4}^{-1} & (e^{d_4}) = e^{-d_4}
\end{align*}
\]
The case for type $D_5$:

By using the orthonormal basis \( \{ v_1, v_2, v_3, v_4, v_5 \} \), the action of the Coxeter element \( \xi_{D_5} = Rf_{d_1}Rf_{d_2}Rf_{d_3}Rf_{d_4}Rf_{d_5} \) is given by

\[
\xi_{D_5} : \begin{cases} 
  v_1 \leftrightarrow -v_1, \\
  v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5 \rightarrow -v_2 \rightarrow -v_3 \rightarrow -v_4 \rightarrow -v_5 \rightarrow v_2.
\end{cases}
\]

Denote \( \eta = e^{2\pi i/8} \) and set \( P \) and \( B \) by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & \eta & i & \eta^3 & 0 & -1 & -\eta & -i & -\eta^3 \\
0 & 1 & i & -1 & -i & 0 & 1 & i & -1 & -i \\
0 & 1 & \eta^3 & -i & \eta & 0 & -1 & -\eta^3 & i & -\eta \\
-2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\
0 & 1 & -\eta & i & -\eta^3 & 0 & -1 & \eta & -i & \eta^3 \\
0 & 1 & -i & -1 & i & 0 & 1 & -i & -1 & i \\
0 & 1 & -\eta^3 & -i & -\eta & 0 & -1 & \eta^3 & i & \eta
\end{pmatrix}
\]

\[
\frac{1}{2\sqrt{2}}
\]

respectively. We define a map \( \sigma_{D_5} \) of \( o_{10}(\mathbb{C}) \) by \( \sigma_{D_5} : A \rightarrow B^{-1}AB \). Then the Coxeter element \( \xi_{D_5} \) is given by \( A \rightarrow P^{-1}AP \) and \( B^{-1}PB \) is a diagonal matrix, \( \text{diag}(-1, 1, \eta, i, \eta^3, 1, -1, -\eta, -i, -\eta^3) \). Note that

\[
^tBB = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

is again a permutation matrix of order 2.
Lemma 4. \( \theta \) and \( \sigma_{D_5} \theta \sigma_{D_5}^{-1} \) commute with each other and

\[
\begin{align*}
\sigma_{D_5} \theta \sigma_{D_5}^{-1}(e^{d_i}) &= -e^{-d_i} & \text{for } i &= 1, 2 \\
\sigma_{D_5} \theta \sigma_{D_5}^{-1}(e^{d_2}) &= -e^{d_1 + d_2 + d_4 + d_5} \\
\sigma_{D_5} \theta \sigma_{D_5}^{-1}(e^{d_3}) &= e^{-d_5} \\
\sigma_{D_5} \theta \sigma_{D_5}^{-1}(e^{d_4}) &= e^{-d_4}
\end{align*}
\]

Summarizing these results, the automorphism \( \xi_L \) and \( \sigma_L^{-1} \theta \sigma_L \) of \( G = (V_L)_1 \) satisfy the following properties:

Proposition 1. Let \( G \) be a simple Lie algebra with a root lattice \( L \) of type \( A_n \) or \( D_n \) \((n = 4, 5)\). Identify \( G \) with the matrix algebra \( sl_{n+1}(\mathbb{C}) \) or \( o_{2n}(\mathbb{C}) \) and define a map \( \theta : A \to -t A \) for \( A \in G \). Let \( T(= \text{the set of all diagonal matrices in } G) \) be a Cartan subalgebra and \( L \) the corresponding root lattice of \( G \). Then there is an automorphism \( \xi_L \) of \( G \) satisfying the following properties:

1. \( \xi_L \) commutes with \( \theta \).
2. The Cartan subalgebra \( T \) is \( \xi_L \)-invariant and the action of \( \xi_L \) on \( T \) coincides with the action of a Coxeter element. In particular, \( \xi_L \) acts on \( T \) fixed point freely.
3. The order of \( \xi_L \) coincides with the Coxeter number (the order of Coxeter element).
4. The action of \( \xi_L \) on the root lattice coincides with a Coxeter element and \( \xi_L \) acts on the set of roots semiregularly. In particular, the length of each orbit is equal to the Coxeter number.
5. The fixed point space \( C = G^{\langle \xi_L \rangle} \) is another Cartan subalgebra on which \( \theta \) acts.
6. There is a matrix \( B \) such that if we define a map \( \sigma_L : A \to B^{-1}AB \), then \( \sigma_L(C) = T \) and \( \sigma_L \theta \sigma_L^{-1} \) satisfies the conclusions in Lemma 2, Lemma 3 and Lemma 4. In particular, \( \sigma_L \theta \sigma_L^{-1} \in \text{Aut}(\hat{L}) \).
7. In the case of \( D_n \), \( \xi_{D_n}(d_1 - d_2) = d_2 - d_1 = 2v_1 \).

We will also call this automorphism \( \xi_L \) of the Lie algebra \( (V_R)_1 \) “a Coxeter element”.

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2.3 Niemeier lattice $N$ and its sublattices

2.3.1 Automorphisms and the dual lattices

When a root lattice $L$ is of type $A_n$, $A^*_n/A_n$ is a cyclic group of order $n + 1$. Denote

$$a[k] = -\frac{1}{n + 1} \left( k \sum_{i=1}^{n+1-k} ia_i + (n + 1 - k) \sum_{i=n-k+2}^n (n + 1 - i)a_i \right)$$

$$= -\frac{1}{n + 1} \left( \sum_{i=1}^{n+1-k} kv_i + \sum_{i=n+2-k}^{n+1} (n + 1 - k)v_i \right)$$

for $k = 1, 2, \ldots, n$. Then $a[1] + A_n$ is a generator of $A^*_n/A_n$ and $a[k] + A_n = ka[1] + A_n$.

Denote the coset $a[k] + A_n$ by $[k]$. Then $A^*_n/A_n = \{[0],[1],\ldots,[n]\} \cong \mathbb{Z}/(n+1)\mathbb{Z}$. Note also that

$$\langle a[k], a[k] \rangle = \frac{k(n + 1 - k)}{n + 1} \quad \text{and} \quad \langle a[k], a_i \rangle = -\delta_{i,n+1-k}.$$ 

Hence, if we ignore the fact that $a[k]$ is not a root, we can still consider the following diagram.

Type A:

```
• − − − − • − − − • − − − − • − − − − •
```

```
a_1 \quad a_2 \quad \ldots \quad a_n \quad a[1]
```

Type D:

```
•|• − − − − • − − − • − − − − • − − − − •
```

```
a[2]
```

```
a_1 \quad a_2 \quad \ldots \quad a_{n-1} \quad a_n
```

Type E:

```
•|• − − − − • − − − • − − − − • − − − − • − − − − •
```

```
a[3] = a[1] + a[2] \quad \text{and} \quad d[2]' = \frac{d_1 - d_2}{2}
```

For the case $L \cong D_4$, $D^*_4/D_4$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Define

$$d[1] = \frac{-2d_2 - d_1 - 2d_3 - d_4}{2}, \quad d[2] = \frac{d_1 + d_2 + 2d_3 + 2d_4}{-2}$$


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For the case $L \cong D_5$, $D_5^*/D_5$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Define
\[
d[1] = \frac{-5d_1 - 3d_2 - 6d_3 - 4d_4 - 2d_5}{4}, \quad d[2] = \frac{d_2 - d_1}{2} = v_1,
\]

In this section, we only consider automorphisms of lattices and so we omit the overline of automorphisms. As we showed, $\overline{\xi}_L$ and $\overline{\theta \sigma_L \theta \sigma_L^{-1}}$ are automorphisms of root lattices $L$. Therefore, we can also extend them to automorphisms of the dual lattice $L^*$. By direct computation, the actions of $\overline{\xi}_L$ and $\overline{\sigma_L \theta \sigma_L^{-1}}$ on $L^*$ are given as follows:

**Lemma 5.** Let $a[k]$ and $d[k]$ be defined as above.
\[
\overline{\xi}_{A_n}(a[1]) = a[1] + (a_1 + \cdots + a_n),
\]
\[
\overline{\sigma_{A_n} \theta \sigma_{A_n}^{-1}}(a[1]) = -a[1],
\]
\[
\overline{\xi}_{D_4}(d[1]) = d[1] + d_1,
\]
\[
\overline{\sigma_{D_4} \theta \sigma_{D_4}^{-1}}(d[1]) = -d[1],
\]
\[
\overline{\xi}_{D_4}(d[2]) = d[2] - (d_1 - d_2),
\]
\[
\overline{\sigma_{D_4} \theta \sigma_{D_4}^{-1}}(d[2]) = -d[2] - d_4,
\]
\[
\overline{\xi}_{D_5}(d[1]) = d[1] + d_1,
\]
\[
\overline{\sigma_{D_5} \theta \sigma_{D_5}^{-1}}(d[1]) = -d[1],
\]
\[
\overline{\xi}_{D_5}(d[2]) = d[2] = d[2] - (d_1 - d_2),
\]
\[
\overline{\sigma_{D_5} \theta \sigma_{D_5}^{-1}}(d[2]) = -d[2] - d_4.
\]

In particular, $\overline{\xi}_L = 1$ and $\overline{\sigma_L \theta \sigma_L^{-1}} = -1$ on the quotient group $L^*/L$ for $L = A_n, D_4,$ and $D_5$.

**[Proof]** We will only prove the first statement. The other cases can be proved similarly. Since $\overline{\sigma \theta \sigma^{-1}}(v_i) = -v_{n+1-i}$ for $i = 1, \ldots, n$ and $\overline{\sigma \theta \sigma^{-1}}(v_{n+1}) = -v_{n+1}$, we have
\[
\overline{\sigma \theta \sigma^{-1}}(a[1]) = \overline{\sigma \theta \sigma^{-1}}\left(\frac{-v_1 + v_2 + \cdots - v_n + n v_{n+1}}{n+2}\right) = \frac{v_1 + v_2 + \cdots + v_n - n v_{n+1}}{n+2} = -a[1].
\]

### 2.3.2 Glue codes and glue vectors

As it is well known, every Niemeier lattice contains a root system of full rank and the Coxeter numbers of all connected components are the same. In other words, if $N$ is a
Niemeier lattice and $\mathcal{R}$ its fundamental root system, then $\mathcal{R}$ has rank 24 and if $\mathcal{R} = \bigcup \mathcal{R}_i$ is a decomposition into a sum of connected components, then the Coxeter number of $\mathcal{R}_i$ is independent of $i$. The Coxeter number of $N$ is defined to be the Coxeter number of a connected component $\mathcal{R}_i$.

Let $R_i = \langle \mathcal{R}_i \rangle$ and $R = \langle \mathcal{R} \rangle$ be the root lattice generated by $\mathcal{R}_i$ and $\mathcal{R}$, respectively. Then $R = \bigoplus R_i$ is a sublattice of $N$. Let $R_i^\vee = \{ v \in \mathbb{Q} \otimes \mathbb{Z} R_i \mid \langle v, u \rangle \in \mathbb{Z} \text{ for all } u \in R_i \}$ be the dual lattice of $R_i$. Then $N \subset \bigoplus R_i^\vee$ and we can consider the quotient group $N/R$ as a subgroup of $\bigoplus R_i^\vee/R_i$. The abelian group $N/R$ is often called a glue code.

If $R_i$ is of type $A_n$, $D_4$, $D_5$, $E_6$, and $E_7$, then $R_i^\vee/R_i$ is isomorphic to $\mathbb{Z}/(n+1)\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, respectively.

We will use the Niemeier lattice of type $A_3$, $A_4$, $A_4D_4$, and $A_2D_2$. A set of generators for the glue code $N/R$ is shown in [CS]. The generators of the glue codes that we will use are as follows:

<table>
<thead>
<tr>
<th>Node $pX$</th>
<th>Type of $N_{pX}$</th>
<th>Coxeter number $q$</th>
<th>Generators for glue code</th>
</tr>
</thead>
<tbody>
<tr>
<td>4A, 4B</td>
<td>$A_2^2D_2^2$</td>
<td>8</td>
<td>[1112], [1721]</td>
</tr>
<tr>
<td>5A</td>
<td>$A_4^6$</td>
<td>5</td>
<td>[1(01441)]</td>
</tr>
<tr>
<td>6A</td>
<td>$A_4^4D_4$</td>
<td>6</td>
<td>[2(024)0], [33001], [30302], [30033]</td>
</tr>
<tr>
<td>3C</td>
<td>$A_3^3$</td>
<td>9</td>
<td>[(114)]</td>
</tr>
</tbody>
</table>

In the above table, if a codeword contains parentheses, this indicates that all vectors obtained by cyclically shifting the part of the vector inside the parentheses are also in the glue code.

Next we explain the glue codes in this list. For each glue codeword $[...j...]$ ($i$-th entry is $j$), an element is defined by setting $(...,a[j],...)$ if $R_i = A_n$ or setting $(...,d[j],...)$ if $R_i = D_4$ is called a glue vector.

For example, in a Niemeier lattice with a root system $A_4^6$,

$$\alpha_{5A} = (0, a[1], 0, a[4], a[3], a[2]) \in (A_4^{\oplus 6})^*$$

is a glue vector for a glue codeword [010432] and $\alpha_{5A} \in N_{A_4^6}$. 40
Let $\xi_{R_i}$ and $\sigma_{R_i} \sigma_{R_i}^{-1}$ be a Coxeter element and an automorphism we defined in the previous section for each $R_i$ and set
\[ \xi = \prod \xi_{R_i} \quad \text{and} \quad \sigma \sigma_{R_i}^{-1} = \prod \sigma_{R_i} \sigma_{R_i}^{-1}. \]
Then $\xi$ and $\sigma \sigma_{R_i}^{-1}$ are automorphisms of $\bigoplus_i R_i$ and $\bigoplus_i R_i^*$. Since $\xi_{R_i} = 1$ and $\sigma_{R_i} \sigma_{R_i}^{-1} = -1$ on $R_i^*/R_i$, $\xi$ and $\sigma \sigma_{R_i}^{-1}$ are also automorphisms of $N$.

2.3.3 $<\xi, \sigma \sigma_{R_i}^{-1}>$-invariant sublattices

The purpose of this subsection is to construct $<\xi, \sigma \sigma_{R_i}^{-1}>$-invariant sublattices $K_{pX}$ ($\cong \sqrt{2}H_{pX}$) and $E_{pX}$ ($\cong \sqrt{2}E_8$) in the Niemeier lattice $N_{pX}$. The method we will use is based on the following observation.

Lemma 6. (1) For an arbitrary root lattice $S$, a sublattice
\[ T_S = \{ (\alpha, \alpha) \mid \alpha \in S \} \subseteq S \oplus S \]
of $S \oplus S$ is $<(\xi_S, \xi_S), (\sigma_S \sigma_S^{-1}, \sigma_S \sigma_S^{-1})>$-invariant and isomorphic to $\sqrt{2}S$.

(2) A sublattice
\[ T_{A_{2n+1}} = \mathbb{Z}(a_1 + a_{n+2}) + \cdots + \mathbb{Z}(a_n + a_{2n+1}) \]
of $A_{2n+1}$ is $<\xi_{A_{2n+1}}, \sigma_{A_{2n+1}} \sigma_{A_{2n+1}}>$-invariant and isomorphic to $\sqrt{2}A_n$. Moreover, on $T_{A_{2n+1}}$, $\xi_{A_{2n+1}}$ coincides with a Coxeter element of $T_{A_{2n+1}}$ as a lattice of type $\sqrt{2}A_n$.

(3) A sublattice
\[ T_{D_n} = \mathbb{Z}(d_2 - d_1) = 2\mathbb{Z}v_1 \]
of $D_n$ is isomorphic to $\sqrt{2}A_1$ and $\xi_{D_n}$ acts on $T_{D_n}$ as $-1$ and $\sigma_{D_n} \sigma_{D_n}^{-1}$ keeps $T_{D_n}$ invariant.

By using the above lemma, we will construct two $<\xi, \sigma \sigma_{R_i}^{-1}>$-invariant lattices $E \cong \sqrt{2}E_8$ and its sublattice $K_{pX} \cong \sqrt{2}H_{pX}$ for each node $pX$ as follows.

4A and 4B case:

We will consider a Niemeier lattice $N_{A_4^2 D_2^2}$ of type $A_4^2 D_2^2$. Let
\[ K_{4A} = \{ (\mathbb{Z}(a_1 + a_5) + \mathbb{Z}(a_2 + a_6) + \mathbb{Z}(a_3 + a_7), 0, \beta, \beta) \mid \beta \in D_5 \} \]
\[ K_{4B} = \{ (\alpha, \alpha, \mathbb{Z}(d_2 - d_1), 0) \mid \alpha \in A_7 \} \]
and define
\[
\alpha_{4A} = (-a_1, -2a_2 - 3a_3 - a_4, -2a_6 - 3a_7, 0, d[3], d[3]) \quad (2, 0, 3, 3)
\] and
\[
\alpha_{4B} = (a[2], a[2], d[2], 0)
\] (2, 2, 2, 0).

Then we have \( K_{4A} \cong \sqrt{2} A_3 + \sqrt{2} D_5 \) and \( K_{4B} \cong \sqrt{2} A_7 + \sqrt{2} A_1 \) and \( E_{4A} = \langle K_{4A}, \alpha_{4A} \rangle \cong E_{4B} = \langle K_{4B}, \alpha_{4B} \rangle \cong \sqrt{2} E_8 \). Furthermore, \( E_{4A}, E_{4B}, K_{4A} \) and \( K_{4B} \) are all \( \langle \xi, \sigma \theta \sigma^{-1} \rangle \)-invariant.

**5A case:**

We will consider a Niemeier lattice \( N_{A_4^6} \) of type \( A_4^6 \). Define
\[
K_{5A} = \{ (0, \alpha, 0, -\alpha, -\beta, \beta) \mid \alpha, \beta \in A_4 \} \cong \sqrt{2} A_4 \oplus \sqrt{2} A_4
\]
\[
\alpha_{5A} = (0, a[1], 0, -a[1], -a[2], a[2])
\]
Then \( K_{5A} \) and \( E_{5A} = \langle H, \alpha_{5A} \rangle \cong \sqrt{2} E_8 \) are \( \langle \xi, \sigma \theta \sigma^{-1} \rangle \)-invariant.

**6A case:**

We will consider a Niemeier lattice \( N_{A_4^2 D_4} \) of type \( A_4^2 D_4 \). In this case, define
\[
K_{6A} = \{ (\alpha, 0, \alpha, Z(a_1 + a_4) + Z(a_2 + a_5), Z(d_2 - d_1)) \mid \alpha \in A_5 \} \cong \sqrt{2} A_3 \oplus \sqrt{2} A_2 \oplus \sqrt{2} A_1
\]
\[
\alpha_{6A} = (a[5], 0, a[5], a[4], d[2]).
\]
Then \( K_{6A} \) and \( E_{6A} = \langle K_{6A}, \alpha_{6A} \rangle \cong \sqrt{2} E_8 \) are \( \langle \xi, \sigma \theta \sigma^{-1} \rangle \)-invariant.

**3C case:**

We will consider a Niemeier lattice \( N_{A_8^3} \) of type \( A_8^3 \). Define
\[
K_{3C} = \{ (-\alpha, 0, \alpha) \mid \alpha \in A_8 \} \cong (\sqrt{2} A_8),
\]
\[
\alpha_{3A} = (-a[3], 0, a[3]).
\]
Then \( K_{3C} \) and \( E_{3C} = \langle K_{3C}, \alpha_{3A} \rangle \cong \sqrt{2} E_8 \) are \( \langle \xi, \sigma \theta \sigma^{-1} \rangle \)-invariant.

It is easy to see the following proposition.

**Proposition 2.** \( \xi \) and \( \overline{\theta} \sigma \theta \sigma^{-1} \) stabilize the cosets in \( E_{pX}/H_{pX} \).
2.4 Automorphisms of $V_N$

2.4.1 $(\hat{J} : e^E)$-automorphism.

In this section, let $E$ denote $E_{pX}$ and set $J = R + E$. Since $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in E$, we may assume that $\iota(E) = \{e^\beta \mid \beta \in E\}$ is a subgroup of $\hat{J} = \{\pm e^\beta \mid \beta \in J\}$. We will call an automorphism $\phi \in \text{Aut}(\hat{J})$ a $(\hat{J} : e^E)$-automorphism if $\phi$ keeps $\{e^\alpha \mid \alpha \in E\}$ invariant. An important property of a $(\hat{J} : e^E)$-automorphism $\phi$ is that it fixes the conformal element $\epsilon_E$ since $\epsilon_E$ is a linear sum of Virasoro element of $V_E$ and $\sum_{\alpha \in E, |\alpha| = 2} e^\alpha$. By direct calculations, we easily have the following lemma.

**Lemma 7.** Assume that $\mu \in \text{Aut}(\hat{R})$ such that $\hat{\mu}(E) = E$ and $\{e^\alpha \mid \alpha \in R \cap E\}$ is $\mu$-invariant. Then the map $\mu_J$ defined by

$$\mu_J(e^\alpha e^\beta) = \mu(e^\alpha)e^{\hat{\mu}(\beta)} \quad \text{for all } \alpha \in R, \beta \in E$$

is a $(\hat{J} : e^E)$-automorphism.

We will next prove the following proposition.

**Proposition 3.** There is an automorphism $\sigma_N$ of $V_N$ satisfying

1. $\sigma_N \theta \sigma_N^{-1}$ is a $(\hat{J} : e^E)$-automorphism,
2. $\sigma_N(\mathcal{C}) = (M(1)^{\otimes n})_1$ for $\mathcal{C} = \oplus_i (V_{R_i})_1^{<\xi_i>}$ and
3. $\sigma_N \theta \sigma_N^{-1} = \bar{\sigma} \theta \bar{\sigma}^{-1}$ on $N$.

**[Proof]** As we showed in Proposition 1, $\sigma_R \theta \sigma_R^{-1} \in \text{Aut}(\hat{R})$ and $\bar{\sigma}_R \theta \bar{\sigma}_R^{-1}$ satisfies the hypothesis in Lemma 7 from our construction of $E$. Therefore we obtain a $(\hat{J}, e^E)$-automorphism $\mu_J$ of $V_J$ from $\sigma_R \theta \sigma_R^{-1}$. Since $\sigma_R$ is an inner automorphism of $(V_{R_1})_1$, there are $w_j \in (V_{R_1})_1$ such that $\sigma_R = \prod_j \exp((w_j)_0)$. Set $\hat{\sigma}_N = \prod_j \exp((w_j)_0) \in \text{Aut}(V_N)$. Then $\sigma_N$ clearly satisfies conditions (2) and (3).

Since $\mu_J^{-1} \sigma_N \theta \sigma_N^{-1}$ fixes all vectors in $(M(1)^{\otimes n})_1$, there is an element $u \in (M(1)^{\otimes n})_1$ such that $\mu_J^{-1} \sigma_N \theta \sigma_N^{-1} = \exp(2\pi i u_0)$ on $V_J$. If $\exp(2\pi i u_0) = 1$, then $\hat{\sigma}_N \theta \hat{\sigma}_N^{-1} = \mu_J$ is a $(\hat{J} : e^E)$-automorphism by the definition of $\mu_J$. So we assume $\exp(2\pi i u_0) \neq 1$. Since $J = R + E = R + \mathbb{Z}\alpha_{pX}$ and $\exp(2\pi i u_0)$ is trivial on $V_R$, the action of $\exp(2\pi i u_0)$ depends only the inner product $\langle \alpha_{pX}, u \rangle \mod \mathbb{Z}$. Therefore, we can choose $u$ in $\mathbb{Z}(0, 0, 0, d[1])$,
\[ \mathbb{Z}(a[1], 0, 0, 0), \mathbb{Z}(0, a[1], 0, 0, 0), \mathbb{Z}(a[1], 0, 0, 0, 0) \text{ and } \mathbb{Z}(a[1], 0, 0) \] for the cases 4A, 4B, 5A, 6A and 3C, respectively. In particular, we have \( \sigma_N \theta \sigma_N^{-1}(u) = \mu_J(u) = -u \).

Set \( \sigma_N = \exp(\pi i u(0)) \sigma_N \), which also satisfies (2) and (3). Then

\[
\sigma_N \theta \sigma_N^{-1} = \exp(\pi i u(0)) \sigma_N \theta \sigma_N^{-1} \exp(-\pi i u(0)) \\
= \exp(\pi i u(0)) \mu_J \exp(2\pi i u(0)) \exp(-\pi i u(0)) \\
= \mu_J \exp(\pi i (\mu_J^{-1}(u) + u)_0) \\
= \mu_J,
\]

which implies that \( \sigma_N \theta \sigma_N^{-1} \) is a \( (\hat{J} : e^F) \)-automorphism as desired. \( \square \)

Similarly, a Coxeter element \( \xi_R = \prod \xi_{R_i} \) of \( (V_N)_1 \) can also be extended to a \( (\hat{J} : e^F) \)-automorphism \( \xi_J \). Next, we will extend \( \xi_J \) to an automorphism \( \xi_N \) of \( V_N \) as follows:

Since \( \sigma((V_R^{\xi_{R^*>}})_1) = (M(1)^{\otimes n})_1, \sigma_N \xi_J \sigma_N^{-1} \) fixes all vectors in \( M(1)^{\otimes n} \) and so there is an element \( u \in (M(1)^{\otimes n})_1 \) such that \( \sigma_N \xi_J \sigma_N^{-1} = \exp(u(0)) \) on \( V_J \). So define

\[
\xi_N = \sigma_N^{-1} \exp(u(0)) \sigma_N \quad \text{on } V_N.
\]

From the definitions of \( \xi_R \) and \( \xi_J \), their orders, \( |\xi_R| \) and \( |\xi_J| \), are the Coxeter number \( q \) of \( N \) and so \( |\xi_N| = qs \) for some integer \( s \). Define \( \phi : N \to \mathbb{Z}/qs\mathbb{Z} \) by

\[
\sigma_N \xi_N \sigma_N^{-1}(e^\alpha) = \exp(2\pi \sqrt{-1} \phi(\alpha)/(qs)) e^\alpha \quad \text{for } \alpha \in N,
\]

where we identify \( \mathbb{Z}/qs\mathbb{Z} \) and \( \mathbb{Z} \mod qs \). Set \( F = \text{Ker} \phi \). We have \( [N : F] = qs \).

**Lemma 8.** \( s = 1, |\xi_N| = q \) and \( F \) is isomorphic to a sublattice of \( \Lambda \) of index \( q \).

**[Proof]** Since \( \xi_R \) is of order \( q \) and \( \text{Ker} \phi \) has no roots, \( \phi(R) \) is a subgroup of \( \mathbb{Z}/qs\mathbb{Z} \) of order \( q \). Hence there is a fundamental root system \( \{x_1, \ldots, x_{24}\} \) satisfying \( \phi(x_i) \equiv s \mod qs \) for all \( i = 1, \ldots, 24 \). Then the highest weight roots \( y_j \) of each connected component also satisfy \( \phi(y_j) \equiv s \mod qs \)

Let \( U = \mathbb{Z} \oplus \mathbb{Z} \) denote a 2-dimensional Lorentzian lattice and set \( \Pi^{25,1} = \Lambda \oplus U \). \( \Pi^{25,1} \) is a 26-dimensional Lorentzian lattice. For each \( \alpha \in \Lambda \), we set \( (\alpha, 1, \langle \alpha, \alpha \rangle/2 - 1) \in \Lambda \oplus U \), which is a root called a Leech root. As it is well known, if a Coxeter number of the root sublattice of a Niemeier lattice \( N \) is \( q \), then there is an isotropic element
\[ \rho_N = (\rho, q, \langle \rho, \rho \rangle/2q) \] in \( \Lambda \oplus U \) such that \( \rho_N^2/\mathbb{Z}\rho_N \) is isomorphic to \( N \) and the set of Leech roots in \( \rho_N^2 \) forms an extended root system of \( R \). Therefore, there are \( \alpha_i \in \Lambda \) such that we can identify the fundamental root system \( \{x_i : i = 1, \ldots, n\} \) with Leech roots \( \{(\alpha_i, 1, \langle \alpha, \alpha \rangle/2 - 1)\} \) in \( \rho_N^2 \). Under this setting, we define

\[ \psi(\beta) = \langle \beta, (0, 0, -s) \rangle \pmod{qs} \] for \( \beta \in N \).

Then \( \phi(\beta) \equiv \psi(\beta) \pmod{qs} \) for \( \beta \in R \). Let \( \alpha \in N \) be an element satisfying \( \phi(\alpha) \equiv 1 \pmod{qs} \), then \( q\alpha \in R \) and so \( q \equiv \phi(q\alpha) \equiv \psi(q\alpha) \equiv qs \pmod{qs} \), which implies \( s = 1 \) as we desired. In particular, we have \( \psi = \phi \) on \( N \). This implies that if \( \beta \in \text{Ker} \phi \), then there is an integer \( k \in \mathbb{Z} \) such that \( \langle \beta, (0, 0, -1) \rangle = kq \). Thus we have

\[ \beta - k\rho_N \in (0, 0, -1)^\bot = \Lambda \oplus \mathbb{Z}(0, 0, 1). \]

Clearly, the embedding of \( \text{Ker} \phi \) into \( \Lambda \) is injective and so \( F \) is isomorphic to a sublattice of the Leech lattice.

\[ \square \]

### 2.5 Proof of the main theorem

By using a sublattice \( K_{pX} \) of \( E \), we can define an automorphism \( \rho_{pX} \) of \( V_E \) as in (1.2).

As we showed, \( \xi_N = 1 \) and \( \sigma_N \theta \sigma_N^{-1} = -1 \) on \( E/K_{pX} \). Hence, we can easily prove the following lemma.

**Lemma 9.** \( \rho_{pX} \) commutes with \( \sigma_N \theta \sigma_N^{-1} \) and \( \xi_N \).

Finally, we will prove the main theorem by using this lemma.

As we showed before, \( \xi_N \) and \( \sigma_N \theta \sigma_N^{-1} \) are \((\hat{J} : e^E)\)-automorphisms. Thus, \( \xi_N \) and \( \sigma_N \theta \sigma_N^{-1} \) both fix the conformal vector \( e_E \) of \( E \). On the other hand, since \( \rho_{pX} \) commutes with \( \xi_N \) and \( \sigma_N \theta \sigma_N^{-1} \), \( \xi_N \) and \( \sigma_N \theta \sigma_N^{-1} \) also fix \( \rho_{pX}(e_E) \). Therefore, \( VA(e_E, \rho_{pX}(e_E)) \subseteq V_N^{<\xi_N, \sigma_N \theta \sigma_N^{-1}>} \)

and we have the desired result:

\[ \sigma_N^{-1}(VA(e_E, \rho_{pX}(e_E))) \subseteq \sigma_N^{-1}(V_N^{<\xi_N, \sigma_N \theta \sigma_N^{-1}>}) \subseteq V_F^{<\theta>} \subseteq V_\Lambda^{<\theta>} \subseteq V^2, \]

which proves our main theorem.
3 Structural theory of Framed VOA

A framed vertex operator algebra $V$ is a simple vertex operator algebra which contains a full sub VOA $F$ called a Virasoro frame isomorphic to a tensor product of $n$-copies of the simple Virasoro VOA $L(\frac{1}{2}, 0)$ such that $\text{rank}(V) = \text{rank}(F) = \frac{n}{2}$. There are many important examples such as the Moonshine VOA $V^\sharp$ and the Leech lattice VOA. In [DGH], a basic theory of framed VOAs was established. A general structure theory about the automorphism group and the frame stabilizer, the subgroup which stabilizes $F$ setwise, was also included. Moreover, Miyamoto [M3] showed that if $V = \oplus_{n \in \mathbb{Z}} V_n$ is a framed VOA over $\mathbb{R}$ such that $V$ has a positive definite invariant bilinear form and $V_1 = 0$, then the full automorphism group Aut($V$) is finite (see also [M1, M2]). Hence, the theory of framed VOA is very useful in studying certain finite groups such as the Monster.

It is well-known (cf. [DMZ, DGH, M3]) that for any framed VOA $V$, one can associate two binary codes $C$ and $D$ to $V$ as follows.

Since $F \simeq L(\frac{1}{2}, 0)^{\otimes n}$ is a rational vertex operator algebra, $V$ is completely reducible as an $F$-module. That is,

$$V = \bigoplus_{h_i \in \{0, 1/2, 1/16\}} m_{h_1, \ldots, h_n} L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_n),$$

where the non-negative integer $m_{h_1, \ldots, h_n}$ is the multiplicity of $L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_n)$ in $V$. In particular, all the multiplicities are finite and $m_{h_1, \ldots, h_n}$ is at most 1 if all $h_i$ are different from 1/16.

Let $M = L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_n)$ be an irreducible module over $F$. The 1/16-word (or $\tau$-word) $\tau(M)$ of $M$ is a binary codeword $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n$ such that

$$\beta_i = \begin{cases} 0 & \text{if } h_i = 0 \text{ or } 1/2, \\ 1 & \text{if } h_i = 1/16. \end{cases} \quad (3.1)$$

For any $\alpha \in \mathbb{Z}_2^n$, define $V^\alpha$ as the sum of all irreducible submodules $M$ of $V$ such that $\tau(M) = \alpha$. Denote $D := \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}$. Then $D$ is an even linear subcode of $\mathbb{Z}_2^n$ and we obtain a $D$-graded structure $V = \bigoplus_{\alpha \in D} V^\alpha$ such that $V^\alpha \cdot V^\beta = V^{\alpha + \beta}$. In particular, $V^0$ forms a subalgebra and $V$ can be viewed as a $D$-graded extension of $V^0$.

For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_2^n$, denote $V(\gamma) = L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_n)$ where $h_i = 1/2$ if $\gamma_i = 1$ and $h_i = 0$ elsewhere. Set

$$C := \{\gamma \in \mathbb{Z}_2^n \mid m_{\gamma_1/2, \ldots, \gamma_n/2} \neq 0\}.$$  

Then $V(0) = F$ and $V^0 = \bigoplus_{\gamma \in C} V(\gamma)$. The sub VOA $V^0$ forms a $C$-graded simple current extension of $F$ which has a unique simple VOA structure. A VOA of the form $V^0 = \bigoplus_{\gamma \in C} V(\gamma)$ is often referred to as a code VOA associated to $C$.

The codes $C$ and $D$ are very important parameters for $V$ and we shall call them the structures codes of $V$ with respect to the frame $F$. One of the main purpose of this paper is to study the precise relations between the structure codes $C$ and $D$. As our main result, we shall show that for any $\alpha \in D$, the subcode $C_\alpha := \{\beta \in C \mid \text{supp}(\beta) \subset \text{supp}(\alpha)\}$ contains a self-dual subcode supporting in $\alpha$ in Theorem 3.31. From this we can prove
that every framed VOA is a $D$-graded simple current extension of a code VOA associated to $C$ in Theorem 3.32. This shows that one can obtain any framed VOA by performing simple current extensions in two steps: first extend $F$ to a code VOA $V_C$ associated to $C$, then form a $D$-graded simple current extension of $V_C$ by adjoining suitable irreducible $V_C$-modules. The structure and representation theory of a simple current extension is well-developed by many authors [?, M3, L3, Y1, Y2]. It is known a simple current extension has a unique structure of a simple vertex operator algebra. Since $F$ is rational, this implies there exist finitely many inequivalent framed VOAs with given central charge. Therefore, together with the conditions on $(C, D)$ in Theorem 3.31, our results provide a method for determining all the framed VOA with a fixed central charge. It is well-known that the structure codes $(C, D)$ of a holomorphic framed VOA must satisfy $C = D^\perp$ (cf. [DGH, M3]). In this case, we will describe necessary and sufficient conditions which $C$ has to satisfy. Namely, we will show in Theorem 3.42 that there exists a holomorphic framed VOA with structure codes $(C, C^\perp)$ if and only if $C$ satisfy the following:

1. The length of $C$ is divisible by 16.
2. $C$ is even, every codeword of $C^\perp$ has a weight divisible by 8, and $C^\perp \subset C$.
3. For any $\alpha \in C^\perp$, the subcode $C_\alpha$ of $C$ contains a doubly even self-dual subcode w.r.t. $\alpha$.

We will call such a code an $F$-admissible code.

Since the conditions above provide quite strong restrictions on a code $C$, it is possible to classify all the codes satisfying these conditions if the length is small. Actually, one can characterize the moonshine vertex operator algebra as the unique holomorphic framed vertex operator algebra of rank 24 with trivial weight one subspace (cf. Remark 3.43). It is a special case of the famous uniqueness conjecture of Frenkel-Lepowsky-Meurman [FLM].

In our argument, doubly even self-dual codes play an important role in prescribing structures of framed VOAs, and it is also revealed that if we omit the doubly even property, then we lose a self-duality of certain summands $V^\alpha$ of $V$. The breaking of self-duality gives rise to an involutive symmetry which is analogous to the lift of the $(-1)$-isometry on a lattice VOA $V_L$. By the standard notation as in [FLM], a lattice VOA has a form

$$V_L = \bigoplus_{\alpha \in L} M_h(\alpha),$$

where $M_h(\alpha)$ is the irreducible highest weight representation over the free bosonic vertex operator algebra $M_h(0)$ associated to the metric space $h = \mathbb{C} \otimes_{\mathbb{Z}} L$ with highest weight $\alpha \in h^* = \mathfrak{h}$. Since the fusion algebra associated to $M_h(0)$ is canonically isomorphic to the group algebra $\mathbb{C}[\mathfrak{h}]$, one has a duality relation $M_h(\alpha)^* \simeq M_h(-\alpha)$. This shows that there exists an order two symmetry inside the decomposition (3.2), namely, we can define an involution $\theta \in \text{Aut}(V_L)$ such that $\theta M_h(\alpha) = M_h(-\alpha)$ which is a lift of an involution on $M_h(0)$. However, since a framed VOA $V$ has a decomposition $V = \bigoplus_{\alpha \in D} V^\alpha$ graded by an elementary abelian 2-group $D$, one cannot see the analogous symmetry directly from the decomposition. We shall show that by breaking the doubly even property in $(C, D)$, we can find a pair of structure subcodes $(C^0, D^0)$ with $[C : C^0] = [D : D^0] = 2$ such that
one can define a decomposition

\[ V = \left( \bigoplus_{\alpha \in D^0} V^{\alpha^+} \oplus V^{\alpha^-} \right) \oplus \left( \bigoplus_{\alpha \in D^1} V^{\alpha^+} \oplus V^{\alpha^-} \right) \tag{3.3} \]

which forms a \((D^0 \oplus \mathbb{Z}_4)\)-graded simple current extension of a code VOA associated to \(C^0\), where \(D^1\) is the complement of \(D^0\) such that \(D = D^0 \sqcup D^1\). Actually, the main motivation of the present work is to obtain the above decomposition. In the study of McKay’s \(E_8\)-observation on the Monster simple group \([\text{LYY1, LYY2}]\), the authors found that the elucidation of the mystery can be reduced to the conjectural \(\mathbb{Z}_p\)-orbifold construction of the moonshine VOA from the Leech lattice VOA for \(p > 2\), where the case \(p = 2\) is solved in \([\text{FLM, Y3}]\). Based on the decomposition (3.3), we will perform a \(\mathbb{Z}_4\)-twisted orbifold construction of \(V^2\).

The order four symmetry defined by the decomposition in (3.3) can be found as an automorphism fixing \(F\) pointwise. The group of automorphisms which fixes \(F\) pointwise is referred to as the pointwise frame stabilizer of \(V\). We shall show that the pointwise frame stabilizer has the order 1, 2 or 4 and it is completely determined by the structure codes \((C, D)\). As an example, we compute the pointwise stabilizer of the Moonshine VOA \(V^\sharp\) associated with a frame given in \([\text{DGH, M3}]\). A \(4A\)-element of the Monster is described as a pointwise frame stabilizer and the associated McKay-Thompson series is computed in the proof of Theorem [3.60]. In addition, the \(4A\)-twisted sector and the \(4A\)-twisted orbifold theory of \(V^2\) are constructed. We shall verify that the top module of this twisted sector is of dimension 1 and of weight 3/4, and the VOA obtained by \(4A\)-twisted orbifold construction of \(V^2\) is isomorphic to \(V^3\) itself.

**Notation and Terminology.** In this article, \(\mathbb{N}, \mathbb{Z}\) and \(\mathbb{C}\) denote the set of non-negative integers, integers, and the complex numbers, respectively. Every vertex operator algebra (VOA for short) is defined over the complex number field \(\mathbb{C}\) unless otherwise stated. A VOA \(V\) is called of \(CFT\)-type if it has the grading \(V = \oplus_{n \geq 0} V_n\) with \(V_0 = \mathbb{C}\cdot 1\). For a VOA structure \((V, Y(\cdot, z), \mathbb{I}, \omega)\) on \(V\), the vector \(\omega\) is called the conformal vector of \(V\). For simplicity, we often use \((V, \omega)\) to denote the structure \((V, Y(\cdot, z), \mathbb{I}, \omega)\). The vertex operator \(Y(a, z)\) of \(a \in V\) is expanded such as \(Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}\). For subsets \(A \subset V\) and \(B \subset M\) of a \(V\)-module \(M\), we set

\[ A \cdot B := \operatorname{Span}_{\mathbb{C}} \{ a(n)v \mid a \in A, v \in B, n \in \mathbb{Z} \}. \]

For \(c, h \in \mathbb{C}\), let \(L(c, h)\) be the irreducible highest weight module over the Virasoro algebra with central charge \(c\) and highest weight \(h\). It is well-known that \(L(c, 0)\) has a simple VOA structure. An element \(e \in V\) is referred to as a Virasoro vector with central charge \(c_e \in \mathbb{C}\) if \(e \in V_2\) and it satisfies \(e(1)e = 2e\) and \(e(3)e = c_e 1\). It is well-known that after setting \(L^c(n) := e_{(n+1)}\), \(n \in \mathbb{Z}\), we obtain a representation of the Virasoro algebra on \(V\) (cf. [M1]), i.e.,

\[ [L^c(m), L^c(n)] = (m - n)L^c(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_e. \]

\(^1\)We have changed the definition of the conformal vector and the Virasoro vector. In our past works, their definitions are opposite.
Therefore, a Virasoro vector together with the vacuum generates a Virasoro VOA inside \( V \). We shall denote this subalgebra by \( \text{Vir}(e) \).

We define a sub VOA of \( V \) to be a pair \((U, e)\) such that \( U \) is a subalgebra of \( V \) containing the vacuum element \( \mathbb{1} \) and \( e \) is the conformal vector of \( U \). Note that \((U, e)\) inherits the grading of \( V \), that is, \( U = \oplus_{n \geq 0} U_n \) with \( U_n = V_n \cap U \), but \( e \) may not be the conformal vector of \( V \). In the case that \( e \) is also the conformal vector of \( V \), we shall call the sub VOA \((U, e)\) a full sub VOA\(^2\).

For a positive definite even lattice \( L \), we shall denote the lattice VOA associated to \( L \) by \( V_L \) (cf. [FLM]). We adopt the standard notation for \( V_L \) as in [FLM]. In particular, \( V_L^+ \) denotes the fixed point subalgebra of \( V_L \) under the lift of \((-1)\)-isometry on \( L \). The letter \( \Lambda \) always denotes the Leech lattice, the unique even unimodular lattice of rank 24 without roots.

Given an automorphism group \( G \) of \( V \), we denote by \( V^G \) the fixed point subalgebra of \( G \) in \( V \). The subalgebra \( V^G \) is called the \( G \)-orbifold of \( V \) in the literature. For a \( V \)-module \((M, Y_M(\cdot, z))\) and \( \sigma \in \text{Aut}(V) \), we set \( Y^\sigma_M(a, z) := Y_M(\sigma a, z) \) for \( a \in V \). Then the \( \sigma \)-conjugate module \( M^\sigma \) of \( M \) is defined to be the module structure \((M, Y^\sigma_M(\cdot, z))\).

We denote the ring \( \mathbb{Z}/p\mathbb{Z} \) by \( \mathbb{Z}_p \) with \( p \in \mathbb{Z} \) and often identify the integers \( 0, 1, \ldots, p-1 \) with their images in \( \mathbb{Z}_p \). An additive subgroup \( C \) of \( \mathbb{Z}_2^n \) together with the standard \( \mathbb{Z}_2 \)-bilinear form is called a linear code. For a codeword \( \alpha = (\alpha_1, \ldots, \alpha_n) \in C \), we define the support of \( \alpha \) by \( \text{supp}(\alpha) := \{i \mid \alpha_i = 1\} \) and the weight by \( \text{wt}(\alpha) := |\text{supp}(\alpha)| \). For a subset \( A \) of \( C \), we define \( \text{supp}(A) := \cup_{\alpha \in A} \text{supp}(\alpha) \). For a binary codeword \( \gamma \in \mathbb{Z}_2^n \) and for any linear code \( C \subset \mathbb{Z}_2^n \), we denote \( C_\gamma := \{\alpha \in C \mid \text{supp}(\alpha) \subset \text{supp}(\gamma)\} \) and \( C^{\perp _\gamma} := \{\beta \in C^{\perp} \mid \text{supp}(\beta) \subset \text{supp}(\gamma)\} \). The all one vector is a codeword \((11 \ldots 1) \in \mathbb{Z}_2^n \).

### 3.1 Preliminaries on simple current extensions

We shall present some basic facts on simple current extensions of a rational \( C_2 \)-cofinite vertex operator algebra of CFT-type.

#### 3.1.1 Fusion algebras

We recall the notion of the fusion algebra associated to a rational VOA \( V \). It is known that a rational VOA \( V \) has finitely many inequivalent irreducible modules (cf. [DLM2]). Let \( \text{Irr}(V) = \{X^i \mid 1 \leq i \leq r\} \) be the set of inequivalent irreducible \( V \)-modules. It is shown in [HL] that the fusion product \( X^i \boxtimes_V X^j \) exists for a rational \( V \). The decomposition of \( X^i \boxtimes_V X^j \) is referred to as the fusion rule:

\[
X^i \boxtimes_V X^j = \bigoplus_{k=1}^r N_{ij}^k X^k,
\]

where the integer \( N_{ij}^k \in \mathbb{Z} \) denotes the multiplicity of \( X^k \) in the fusion product, and is called the fusion coefficient which is also the dimension of the space of all \( V \)-intertwining operators of type \( X^i \times X^j \to X^k \). We shall denote by \((X^k_{X^i X^j})_V \) the space of \( V \)-intertwining operators of type \( X^i \times X^j \to X^k \). We define the fusion algebra (or the Verlinde algebra)

\(^2\)It is also called a conformal sub VOA in the literature.
associated to $V$ by the linear space $\mathcal{V}(V) = \bigoplus_{i=1}^{r} \mathbb{C}X^i$ spanned by a formal basis $\{X^i \mid 1 \leq i \leq r\}$ equipped with a product defined by the fusion rule (3.4). By the symmetry of fusion coefficients, the fusion algebra $\mathcal{V}(V)$ is commutative (cf. [FHL]). Moreover, it is shown in [H2] that if $V$ is rational, $C_2$-cofinite and of CFT-type, then $\mathcal{V}(V)$ is associative.

In this subsection, we assume that $V$ is rational, $C_2$-cofinite and of CFT-type.

A $V$-module $M$ is called a **simple current** if for any irreducible $V$-module $X$, the fusion product $M \boxtimes_V X$ is also irreducible. In other words, a simple current $V$-module $M$ induces a permutation on $\text{Irr}(V)$ via $X \mapsto M \boxtimes_V X$ for $X \in \text{Irr}(V)$. Note that $V$ itself is a simple current $V$-module.

Next we shall recall the notion of the dual (or contragredient) module. For a graded $V$-module $M = \bigoplus_{n \in \mathbb{N}} M_{n+h}$ such that $\dim M_{n+h} < \infty$, define its restricted dual by $M^* = \bigoplus_{n \in \mathbb{N}} M^*_{n+h}$, where $M^*_{n+h} := \text{Hom}_V(M_{n+h}, \mathbb{C})$ is the dual space of $M_{h+n}$. Let $Y_M(a, z)$ be the vertex operator of $M$. Then we can introduce a $V$-module structure on $M^*$ with a vertex operator $Y_M^*(a, z)$ defined by

$$
(Y_M^*(a, z)x, v) := \langle x, Y_M(e^{zL(1)}(-z^{-2})L(0))a, z^{-1})v \rangle
$$

(3.5) for $a \in V$, $x \in M^*$ and $v \in M$ (cf. [FHL]).

Note that if the dual module $M^*$ of $M$ is isomorphic to $N$, then there exists a $V$-isomorphism $f \in \text{Hom}_V(N, M^*)$. Then $f$ induces a $V$-intertwining operator of type $V \times N \rightarrow M$. This implies that $(Y_M^*)^{-1} \neq 0$ or equivalently $M \boxtimes_V N \supset V^*$. A $V$-module $M$ is called **self-dual** if $M^* \simeq M$. It is obvious that the space of $V$-invariant bilinear forms on an irreducible self-dual $V$-module is one-dimensional.

**Lemma 3.1.** ([Y2]) **Let $U, W$ be $V$-modules such that $U \boxtimes_V W = V$. Then both $U$ and $W$ are simple current $V$-modules.**

**Corollary 3.2.** Assume that $V$ is simple, rational, $C_2$-cofinite, of CFT-type and self-dual. Then the followings hold.

1. Every simple current $V$-module is irreducible.
2. A $V$-module $U$ is simple current if and only if $U \boxtimes_V U^* = V$.
3. The set of simple current $V$-modules forms a multiplicative abelian group in $\mathcal{V}(V)$ under the fusion product.

**Proof:** Let $U$ be a simple current $V$-module. Then $U = V \boxtimes_V U$ is irreducible as $V$ is simple. By the symmetry of fusion rules $(V^U)_V \simeq (U^U)_V \simeq (U^V)_V$ (cf. [FHL]) and $V^* \simeq V$, we have $U \boxtimes V U^* \supset V$. Since $U$ is irreducible, we have $U \boxtimes V U^* = V$. This shows (1) and (2). Now let $A$ be the subset of $\mathcal{V}(V)$ consisting of all the (inequivalent) simple current $V$-modules. Since a fusion product of simple current modules is again a simple current, $A$ is closed under the fusion product. Clearly $V \in A$ so that $A$ contains a unit element. Finally, if $U \in A$, then $U \boxtimes V U^* = V$ so that the inverse $U^* \in A$ by (2). This completes the proof.

**3.1.2 Simple current extensions**

We recall the notion of simple current extensions from [Y1]. Let $V^0$ be a simple rational $C_2$-cofinite VOA of CFT type and let $\{V^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible
Theorem 3.4. Let $\mathcal{M}(Y_1)$ be a $D$-graded extension of $V^0$ if $V^0$ is a full sub VOA of $V_D$ and $V_D$ carries a $D$-grading, i.e., $Y(x^\alpha, z)x^\beta \in V^{\alpha + \beta}$ for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. In this case, the dual group $D^*$ of $D$ acts naturally and faithfully on $V_D$. If all $V^\alpha$, $\alpha \in D$, are simple current $V^0$-modules, then $V_D$ is referred to as a $D$-graded simple current extension of $V^0$. The abelian group $D$ is automatically finite since $V^0$ is rational (cf. [DLM2]). We shall list some basic facts about simple current extensions.

Proposition 3.3. ([ABD, DME, L3, Y1]) Let $V^0$ be a simple rational $C_2$-cofinite VOA of CFT type. Let $V_D = \oplus_{\alpha \in D} V^\alpha$ be a $D$-graded simple current extension of $V^0$. Then

1. $V_D$ is rational and $C_2$-cofinite.
2. If $\tilde{V}_D = \oplus_{\alpha \in D} \tilde{V}^\alpha$ is another $D$-graded simple current extension of $V^0$ such that $\tilde{V}^\alpha \simeq V^\alpha$ as $V^0$-modules, then $V_D$ and $\tilde{V}_D$ are isomorphic VOAs over $\mathbb{C}$. One can define a VOA isomorphism between $V_D$ and $\tilde{V}_D$ by a $V^0$-isomorphism.
3. For any subgroup $E$ of $D$, a subalgebra $V_E := \oplus_{\alpha \in E} V^\alpha$ is an $E$-graded simple current extension of $V^0$. Moreover, $V_D$ is a $D/E$-graded simple current extension of $V_E$.

A representation theory of simple current extensions is developed in [L3, Y1]. It is shown that each irreducible module over a simple current extension corresponds to an irreducible module over a finite dimensional semisimple associative algebra. Moreover, it is also proved that any $V^0$-module can be extended to a certain twisted module over $V_D$. We review some results on representations from [L3, Y1].

Let $M$ be an irreducible $V_D$-module. We can take an irreducible $V^0$-submodule $W$ of $M$ since $V^0$ is rational. Define $D_W := \{ \alpha \in D \mid V^\alpha \boxtimes_{V^0} W \simeq_{V^0} W \}$. Then $D_W$ forms a subgroup of $D$ and $D_W = D_W'$ for any irreducible $V^0$-submodule $W'$ of $M$. We call $M$ $D$-stable if $D_W = 0$. In this case, $V^\alpha \boxtimes_{V^0} W \simeq_{V^0} V^\beta \boxtimes_{V^0} W$ if and only if $\alpha = \beta$ and by setting $M^\alpha := V^\alpha \boxtimes_{V^0} W$, we have a $D$-graded isotypical decomposition $M = \oplus_{\alpha \in D} M^\alpha$ as a $V^0$-module.

Theorem 3.4. ([L3, Y1]) Let $W$ be an irreducible $V^0$-module. Then there exists a unique $\chi_W \in D^* \subset \text{Aut}(V_D)$ such that $W$ can be extended to an irreducible $\chi_W$-twisted $V_D$-module. If $D_W = 0$, then $W$ is uniquely extended to an irreducible $D$-stable $\chi_W$-twisted $V_D$-module which is given by $V_D \boxtimes_{V^0} W$ as a $V^0$-module.

One can easily compute fusion rules among irreducible $D$-stable modules.

Proposition 3.5. ([Y1]) Let $V_D$ be a $D$-graded simple current extension of a simple rational $C_2$-cofinite VOA $V^0$ of CFT-type. Let $M^i$, $i = 1, 2, 3$ be irreducible $D$-stable $V_D$-modules. Denote by $M^i = \oplus_{\alpha \in D}(M^i)^\alpha$ a $D$-graded isotypical decomposition of $M^i$. Then the following linear isomorphism holds:

$$
\begin{pmatrix}
M^3 \\
M^1 \\
M^2
\end{pmatrix}
\bigg|_{V_D} \simeq
\begin{pmatrix}
(M^3)^\gamma \\
(M^1)^\alpha \\
(M^2)^\beta
\end{pmatrix}
\bigg|_{V^0},
$$

where $\alpha, \beta, \gamma \in D$ are arbitrary.

We will need the following result on a $\mathbb{Z}_2$-graded simple current extension.
Proposition 3.6. Let \( V^0 \) be a simple rational \( C_2 \)-cofinite self-dual VOA of CFT-type. Let \( V^1 \) be a simple current \( V^0 \)-module not isomorphic to \( V^0 \) such that \( V^1 \otimes_{V^0} V^1 = V^0 \). Assume that \( V^1 \) has an integral top weight and the invariant bilinear form on \( V^1 \) is symmetric. Then there exists a unique simple VOA structure on \( V = V^0 \oplus V^1 \) as a \( \mathbb{Z}_2 \)-graded simple current extension of \( V^0 \).

Later, we shall consider a construction of framed VOAs. The following extension property will be used frequently.

Theorem 3.7. (Extension property [Y2, Theorem 4.6.1]) Let \( V^{(0,0)} \) be a simple rational \( C_2 \)-cofinite VOA of CFT-type, and let \( D_1, D_2 \) be finite abelian groups. Assume that we have a set of inequivalent irreducible simple current \( V^{(0,0)} \)-modules \( \{ V^{(\alpha,\beta)} \mid (\alpha, \beta) \in D_1 \oplus D_2 \} \) with \( D_1 \oplus D_2 \)-graded fusion rules \( V^{(\alpha_1,\beta_1)} \otimes_{V^{(0,0)}} V^{(\alpha_2,\beta_2)} = V^{(\alpha_1+\alpha_2,\beta_1+\beta_2)} \) for any \( (\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_1 \oplus D_2 \). Further assume that all \( V^{(\alpha,\beta)} \), \( (\alpha, \beta) \in D_1 \oplus D_2 \), have integral top weights and we have \( D_1 \)- and \( D_2 \)-graded simple current extensions \( V_{D_1} = \oplus_{\alpha \in D_1} V^{(\alpha,0)} \) and \( V_{D_2} = \oplus_{\beta \in D_2} V^{(0,\beta)} \). Then \( V_{D_1 \oplus D_2} := \oplus_{(\alpha,\beta) \in D_1 \oplus D_2} V^{(\alpha,\beta)} \) possesses a unique structure of a simple vertex operator algebra as a \( (D_1 \oplus D_2) \)-graded simple current extension of \( V^{(0,0)} \).

3.2 Ising frame and framed VOA

We shall review the notion of an Ising frame and a framed vertex operator algebra.

3.2.1 Miyamoto involutions

We begin by the definition of an Ising vector.

Definition 3.8. A Virasoro vector \( e \) is called an Ising vector if \( \text{Vir}(e) \simeq L(1/2, 0) \). Two Virasoro vectors \( u, v \in V \) are called orthogonal if \( [Y(u, z_1), Y(v, z_2)] = 0 \). A decomposition \( \omega = e^1 + \cdots + e^n \) of the conformal vector \( \omega \) of \( V \) is called orthogonal if \( e^i \) are mutually orthogonal Virasoro vectors.

Let \( e \in V \) be an Ising vector. By definition, \( \text{Vir}(e) \simeq L(1/2, 0) \). It is well-known that \( L(1/2, 0) \) is rational \( C_2 \)-cofinite and has three irreducible representations \( L(1/2, 0), L(1/2, 1/2) \) and \( L(1/2, 1/16) \). The fusion rules of \( L(1/2, 0) \)-modules are computed in [DMZ]:

\[
L(1/2, 1/2) \otimes L(1/2, 1/2) = L(1/2, 0), \quad L(1/2, 1/2) \otimes L(1/2, 1/16) = L(1/2, 1/16),
L(1/2, 1/16) \otimes L(1/2, 1/16) = L(1/2, 0) \oplus L(1/2, 1/2).
\]

(3.6)

By (3.6), one can define some involutions in the following way. Let \( V_e(h) \) be the sum of all irreducible \( \text{Vir}(e) \)-submodules of \( V \) isomorphic to \( L(1/2, h) \) for \( h = 0, 1/2, 1/16 \). Then one has the isotypical decomposition:

\[
V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16).
\]

Define a linear automorphism \( \tau_e \) on \( V \) by

\[
\tau_e = \begin{cases} 
1 & \text{on } V_e(0) \oplus V_e(1/2), \\
-1 & \text{on } V_e(1/16).
\end{cases}
\]

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Then by the fusion rules in \(\text{(3.6)}\), \(\tau_e\) defines an automorphism on the VOA \(V\) (cf. [M1]). On the fixed point subalgebra \(V^{(\tau_e)} = V_e(0) \oplus V_{e}(1/2)\), one can define another linear automorphism \(\sigma_e\) by
\[
\sigma_e = \begin{cases} 
1 & \text{on } V_e(0), \\
-1 & \text{on } V_e(1/2).
\end{cases}
\]
Then \(\sigma_e\) also defines an automorphism on \(V^{(\tau_e)}\) (cf. [M1]). The automorphisms \(\tau_e \in \text{Aut}(V)\) and \(\sigma_e \in \text{Aut}(V^{(\tau_e)})\) are often called Miyamoto involutions.

### 3.2.2 Framed VOAs and their structure codes

**Definition 3.9.** ([DCHL][M3]) A simple vertex operator algebra \((V, \omega)\) is called framed if there exists a mutually orthogonal set \(\{e^1, \ldots, e^n\}\) of Ising vectors such that \(\omega = e^1 + \cdots + e^n\) is an orthogonal decomposition. The full sub VOA \(F\) generated by \(e^1, \ldots, e^n\) is called an Ising frame or simply a frame of \(V\).

Let \((V, \omega)\) be a framed VOA with an Ising frame \(F\). Then
\[
F \cong \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n) \cong L(1/2,0)^{\otimes n}
\]
and \(V\) is a direct sum of irreducible \(F\)-submodules \(\otimes_{i=1}^{n} L(1/2, h_i)\) with \(h_i \in \{0, 1/2, 1/16\}\). For each irreducible \(F\)-module \(\otimes_{i=1}^{n} L(1/2, h_i)\), we define its binary 1/16-word (or \(\tau\)-word) \((\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_2^n\) by \(\alpha_i = 1\) if and only if \(h_i = 1/16\). For \(\alpha \in \mathbb{Z}_2^n\), denote by \(V^\alpha\) the sum of all irreducible \(F\)-submodules of \(V\) whose 1/16-words are equal to \(\alpha\). Define a linear code \(D \subset \mathbb{Z}_2^n\) by \(D = \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}\). Then we have the \(1/16\)-word decomposition \(V = \oplus_{\alpha \in D} V^\alpha\). By the fusion rules of \(L(1/2,0)\)-modules, it is easy to see that \(V^\alpha \cdot V^\beta \subset V^{\alpha + \beta}\). Hence, the dual group \(D^*\) of \(D\) acts on \(V\). In fact, the action of \(D^*\) coincides with that of the elementary abelian 2-group generated by Miyamoto involutions \(\{\tau_e \mid 1 \leq i \leq n\}\). Therefore, all \(V^\alpha, \alpha \in D\), are irreducible \(V^0\)-modules by \([?]\). Since there is no \(L(1/2,1/16)\)-component in \(V^0\), the fixed point subalgebra \(V^{D^*} = V^0\) has the following shape:
\[
V^0 = \bigoplus_{h_i \in \{0,1/2\}} m_{h_1, \ldots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),
\]
where \(m_{h_1, \ldots, h_n} \in \mathbb{N}\) denotes the multiplicity. On \(V^0\) we can define Miyamato involutions \(\sigma_e^i\) for \(i = 1, \ldots, n\). Denote by \(Q\) the elementary abelian 2-subgroup of \(\text{Aut}(V^0)\) generated by \(\{\sigma_e^i \mid 1 \leq i \leq n\}\). Then \((V^0)^Q = F\) and each \(m_{h_1, \ldots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)\) is an irreducible \(F\)-submodule again by \([?]\). Thus \(m_{h_1, \ldots, h_n} \in \{0, 1\}\) and we obtain an even linear code \(C := \{(2h_1, \ldots, 2h_n) \in \mathbb{Z}_2^n \mid h_i \in \{0,1/2\}, \ m_{h_1, \ldots, h_n} \neq 0\}\), namely,
\[
V^0 = \bigoplus_{\alpha = (\alpha_1, \ldots, \alpha_n) \in C} L(1/2, \alpha_1/2) \otimes \cdots \otimes L(1/2, \alpha_n/2). \tag{3.7}
\]
Since \(L(1/2,0)\) and \(L(1/2,1/2)\) are simple current \(L(1/2,0)\)-modules, \(V^0\) is a \(C\)-graded simple current extension of \(F\). By Proposition [3.3] the VOA structure on \(V^0\) is unique. The
VOA $V^0$ of the form (3.7) is called the code VOA associated to $C$ and denoted by $V_C$. It is clear that $V_C$ is simple, rational, $C_2$-cofinite, self-dual and of CFT-type, and hence $V$ is also self-dual (cf. [Li1]).

Summarizing, there exists a pair $(C, D)$ of even linear codes such that $V$ is an $D$-graded extension of a code VOA $V_C$ associated to $C$. We call the pair $(C, D)$ the structure codes of a framed VOA $V$ associated with the frame $F$. Since the powers of $z$ in an $L(1/2, 0)$-intertwining operator of type $L(1/2, 1/2) \times L(1/2, 1/2) \to L(1/2, 1/16)$ are half-integral, the structure codes $(C, D)$ satisfy $C \subset D^\perp$.

**Notation.** Let $V$ be a framed VOA with the structure codes $(C, D)$, $C, D < \mathbb{Z}_2^n$. For a binary codeword $\beta \in \mathbb{Z}_2^n$, we define:

$$\sigma_\beta := \prod_{i \in \text{supp}(\beta)} \sigma_{e^i} \in \text{Aut}(V^0) \quad \text{and} \quad \tau_\beta := \prod_{i \in \text{supp}(\beta)} \tau_{e^i} \in \text{Aut}(V).$$

(3.8)

Namely, by associating Miyamoto involutions to a codeword of $\mathbb{Z}_2^n$, $\sigma : \mathbb{Z}_2^n \to \text{Aut}(V^0)$ and $\tau : \mathbb{Z}_2^n \to \text{Aut}(V)$ are group homomorphisms. It is also clear that $\ker \sigma = C^\perp$ and $\ker \tau = D^\perp$.

### 3.3 Representation of code VOAs

Since every framed VOA is an extension of its code sub VOA, it is quite standard to study a framed VOA as a module over its code VOA. Certain fusion rules of modules over a code VOA prescribe the structure of a framed VOA. In order to compute the fusion rules, we review a structure theory of the representations of a code VOA.

#### 3.3.1 Central extension of codes

Let $\nu^1 = (10 \ldots 0)$, $\nu^2 = (010 \ldots 0)$, ..., $\nu^n = (0 \ldots 01) \in \mathbb{Z}_2^n$. Set $\varepsilon : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{C}^\ast$ by

$$\varepsilon(\nu^i, \nu^j) := -1 \quad \text{if} \quad i > j \quad \text{and} \quad 1 \quad \text{otherwise},$$

(3.9)

and extend to $\mathbb{Z}_2^n$ linearly. Then $\varepsilon$ defines a 2-cocycle in $Z^2(\mathbb{Z}_2^n, \mathbb{C}^\ast)$. By definition,

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \quad \text{and} \quad \varepsilon(\alpha, \alpha) = (-1)^{\langle \alpha, \alpha \rangle (\langle \alpha, \alpha \rangle - 1)/2}$$

(3.10)

for all $\alpha, \beta \in \mathbb{Z}_2^n$. In particular, $\varepsilon(\alpha, \alpha) = (-1)^{\langle \alpha, \alpha \rangle / 2}$ if $\alpha \in \mathbb{Z}_2^n$ is even.

Let $G$ be the central extension of $\mathbb{Z}_2^n$ by $\mathbb{C}^\ast$ with associated 2-cocycle $\varepsilon$. Recall that $G = \mathbb{Z}_2^n \times \mathbb{C}^\ast$ as a set, but has a group operation by

$$(\alpha, u)(\beta, v) = (\alpha\beta, \varepsilon(\alpha, \beta)uv)$$

for all $\alpha, \beta \in \mathbb{Z}_2^n$ and $u, v \in \mathbb{C}^\ast$. Let $C$ be a binary even linear code of $\mathbb{Z}_2^n$. Since $\varepsilon$ takes values in $\{\pm 1\}$, we can take a subgroup $\tilde{C} = \{(\alpha, u) \in G \mid \alpha \in C, u \in \{\pm 1\}\}$ of $G$ which forms a central extension of $C$ by $\{\pm 1\}$:

$$1 \to \{\pm 1\} \to \tilde{C} \xrightarrow{\pi} C \to 1.$$  

(3.11)
Take a subgroup $D$ of $C$ such that $C = (C \cap C^\perp) \oplus D$. By (3.10), the preimage $\pi^{-1}(C \cap C^\perp)$ is the center of $\tilde{C}$. Since the pairing $(\cdot, \cdot)$ is non-degenerate on $D$, it follows from (3.10) that the preimage $\tilde{D} := \pi^{-1}(D)$ is a extra-special 2-subgroup of $\tilde{C}$. Therefore, $C$ is isomorphic to a central product of $\tilde{D}$ and $\pi^{-1}(C \cap C^\perp)$ over $\{\pm 1\}$ which we shall denote by $\tilde{D} *_{\{\pm 1\}} \pi^{-1}(C \cap C^\perp)$.

Let $\mathbb{C}^*\tilde{C} = \{(\alpha, u) \in G \mid \alpha \in C, u \in \mathbb{C}^*\}$ be a subgroup of $G$ containing $\tilde{C}$. Then we have the following exact sequence:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^*\tilde{C} \overset{\pi_{\mathbb{C}^*}}{\longrightarrow} C \longrightarrow 1. \quad (3.12)$$

Since the preimage of $C \cap C^\perp$ in $\mathbb{C}^*\tilde{C}$ splits, one has an isomorphism

$$\mathbb{C}^*\tilde{C} \simeq (C \cap C^\perp) \times (\mathbb{C}^* *_{\{\pm 1\}} \tilde{D}).$$

Now let $\psi : C \to \text{End}(V)$ be a $\varepsilon$-projective representation of $C$ on $V$, that is, $\psi(\alpha)\psi(\beta) = \varepsilon(\alpha, \beta)\psi(\alpha + \beta)$ for $\alpha, \beta \in C$. Then one defines a linear representation $\tilde{\psi}$ of $\mathbb{C}^*\tilde{C}$ via $\tilde{\psi}(\tilde{\alpha}, \tilde{u}) := w\psi(\alpha) \in \text{End}(V)$ for $\alpha \in C$ and $u \in \mathbb{C}^*$. Since $\mathbb{C}^*\tilde{C}$ is isomorphic to a direct product of $C \cap C^\perp$ and $\mathbb{C}^* *_{\{\pm 1\}} \tilde{D}$, $\tilde{\psi}$ is a tensor product of a linear character of $C \cap C^\perp$ and an irreducible non-linear representation of $\tilde{D}$ if $\tilde{\psi}$ is irreducible. It is well-known that the irreducible non-linear representation of $\tilde{D}$ is unique up to isomorphism (cf. [FLM]). Therefore, the number of inequivalent irreducible $\varepsilon$-projective representation of $\tilde{C}$ is equal to the order of $C \cap C^\perp$. Let us review the structure of the irreducible non-linear representation of $\tilde{D}$ in a bit detail. Let $H$ be a maximal self-orthogonal subcode of $D$. Then by (3.10) the preimage $\pi_{\mathbb{C}^*}^{-1}(H)$ of $H$ in $\mathbb{C}^*\tilde{C}$ splits so that we can take a section map $\iota : H \to \mathbb{C}^*$ such that $H \ni \alpha \mapsto (\alpha, \iota(\alpha)) \in \pi_{\mathbb{C}^*}^{-1}(H)$ is a group homomorphism. Let $\chi$ be a linear character of $H$ and define a linear character $\tilde{\chi}$ of $\pi_{\mathbb{C}^*}^{-1}(H)$ by $\tilde{\chi}(\alpha, \iota(\alpha)u) = u\chi(\alpha)$ for $\alpha \in H$ and $u \in \mathbb{C}^*$. Since the preimage $\tilde{H} := \pi^{-1}(H)$ is a subgroup of $\pi_{\mathbb{C}^*}^{-1}(H)$, we may consider $\tilde{\chi}$ as a linear character of $\tilde{H}$. Then the irreducible non-linear representation of $\tilde{D}$ is realized by the induced module $\text{Ind}^{\tilde{D}}_{\tilde{H}}\tilde{\chi}$ by Theorem 5.5.1 of [FLM]. Summarizing, we have:

**Proposition 3.10.** (Theorem 5.5.1 of [FLM]) Let $\psi$ be an irreducible $\varepsilon$-projective representation of $C$. Then the associated linear representation $\tilde{\psi}$ of $\mathbb{C}^*\tilde{C}$ is of the form $\lambda \otimes_{\mathbb{C}} \text{Ind}^{\tilde{D}}_{\tilde{H}}\tilde{\chi}$, where $\lambda$ is a linear character of $C \cap C^\perp$, $\tilde{H}$ is the preimage of a maximal self-orthogonal subcode $H$ of $D$ in $\tilde{C}$, and $\tilde{\chi}$ is a linear character of $\tilde{H}$ such that $\tilde{\chi}(0, -1) = -1$. In particular, $\tilde{\psi}$ is induced from a linear character of a maximal abelian subgroup of $\tilde{C}$.

### 3.3.2 Structure of modules

Let $C$ be an even linear code of $\mathbb{Z}_2^n$. For $h_i \in \{0, 1/2, 1/16\}$, we shall denote

$$L(h_1, \ldots, h_n) := L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_n).$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$, we set

$$V(\alpha) := L(\alpha_1/2, \ldots, \alpha_n/2).$$
Let $V_C = \bigoplus_{\alpha \in C} V(\alpha)$ be the code VOA associated to $C$. Since $V(0) = L(1/2,0)^{\otimes n}$ is a rational full sub VOA of $V_C$, every $V_C$-module is completely reducible as a $V(0)$-module. We shall review the structure theory of irreducible $V_C$-modules from $M3$, $L3$, $Y1$, $Y2$.

Let $M$ be an irreducible $V_C$-module. Take an irreducible $V(0)$-submodule $W$ of $M$, which is possible as $V(0)$ is rational. Let $\tau(W) \in Z_2^*$ be the binary $1/16$-word of $W$. Then it follows from the fusion rules of $L(1/2,0)$-modules that $\tau(W) \in C^+$ and $\tau(W) = \tau(W')$ for any irreducible $V(0)$-submodule $W'$ of $M$. Set $C_W := \{ \alpha \in C \mid V^\alpha \cong W \}$. Then $C_W = \{ \alpha \in C \mid \text{supp}(\alpha) \subset \text{supp}(\tau(W)) \}$. Let $\{ \alpha_i \mid 1 \leq i \leq r \}$ be the coset representatives of $C/C_W$. Then $V(\alpha_i) \cong_W V(\alpha_j)$ if $i \neq j$. For simplicity, we set $W^i := V(\alpha_i) \cong W$. Then we have the following isotypical decomposition:

$$M = \bigoplus_{i=1}^r W^i \otimes \text{Hom}(V(0), \tau(W^i), M).$$

Each homogeneous component $W^i \otimes \text{Hom}(V(0), \tau(W^i), M)$ is an irreducible $C_{\tilde{W}}$-submodule, where $C_{\tilde{W}}$ is a code VOA associated to $\tilde{C}_W$. Let $U := \text{Hom}(V(0), W, M)$. It is shown in $M3$, $L3$, $Y1$ that $U$ is an irreducible $\varepsilon$-projective representation of $C_W$ so that $U$ is also an irreducible $\mathbb{C}^*\tilde{C}_W$-module. Moreover, the $V_C$-module structure on $M$ is uniquely determined by the $\mathbb{C}^*\tilde{C}_W$-module structure on $U$.

**Theorem 3.11.** (M3, L3, Y1) Let $C$ be an even linear code and $V_C = \bigoplus_{\alpha \in C} V(\alpha)$ the associated code VOA. Let $W$ be an irreducible $V^0$-module such that $\tau_W \in C^+$. Then there is a one to one correspondence between the isomorphism classes of irreducible $\varepsilon$-projective representations of $C_W$ and the isomorphism classes of irreducible $V_C$-modules containing $W$ as a $V^0$-submodule.

In the following, we shall give an explicit construction of irreducible $V_C$-modules from irreducible $\varepsilon$-projective $C_W$-modules.

**An Explicit construction.** Let $W$ be an irreducible $V(0)$-module such that the $1/16$-word $\tau(W) \in C^+$. Let $H$ be a maximal self-orthogonal subcode of $C_W = \{ \alpha \in C \mid \text{supp}(\alpha) \subset \text{supp}(\tau(W)) \}$. Since the preimage $\pi_{C_W}^{-1}(H)$ of $H$ in (3.12) splits, there is a section map $\iota : H \to \mathbb{C}^*$ such that $\langle \alpha, \iota(\alpha) \rangle(\beta, \iota(\beta)) = (\alpha + \beta, \iota(\alpha + \beta))$ for all $\alpha, \beta \in H$. Let $\chi$ be a linear character of $H$. Then we can define a linear character $\tilde{\chi}$ of $\pi_{C_W}^{-1}(H)$ by

$$\tilde{\chi}(\alpha, \iota(\alpha)u) = u\chi(\alpha) \quad \text{for} \quad \alpha \in H, \ u \in \mathbb{C}^*.$$  

(3.13)

In this case, $\tilde{\chi}$ is also a linear character on the preimage $\tilde{H} = \pi^{-1}(H)$ of $H$ in (3.11). Let $C^*[C]$ be the twisted group algebra associated to the 2-cocycle $\varepsilon \in Z^2(C, \mathbb{C}^*)$ defined in (3.9). That means $C^*[C] = \text{Span}_C \{ e^\alpha \mid \alpha \in C \}$ as a linear space and $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$. By (3.10), we have

$$e^\alpha e^\beta = (-1)^{\langle \alpha, \beta \rangle} e^{\beta} e^\alpha.$$  

(3.14)

It is clear that $C^*[C_W] = \bigoplus_{\alpha \in C_W} \mathbb{C} e^\alpha$ and $C^*[H] = \bigoplus_{\beta \in H} \mathbb{C} e^\beta$ are subalgebras of $C^*[C]$. Moreover, $C^*[H] \simeq \mathbb{C}[\tilde{H}] \simeq \mathbb{C}[H]$ as $\mathbb{C}$-algebras. Let $\{ \alpha_1, \ldots, \alpha_r \}$ be a set of coset
representatives of $C/C_W$ and let $\{\beta_1, \ldots, \beta_s\}$ be a set of coset representatives of $C_W/H$. Consider an induced module $\text{Ind}_H^\mathbb{C} \bar{\chi}$. As a linear space, it is defined by

$$\text{Ind}_H^\mathbb{C} \bar{\chi} = \bigoplus_{i=1}^r \bigoplus_{j=1}^s \mathbb{C} e^{\alpha_i + \beta_j} \otimes v_{\bar{\chi}},$$

where $\mathbb{C} v_{\bar{\chi}}$ is a $\mathbb{C}[H]$-module affording the character $\bar{\chi}$, that is, $\iota(\alpha) e^\alpha \cdot v_{\bar{\chi}} = \chi(\alpha) v_{\bar{\chi}}$ for all $\alpha \in H$. Note also that the component

$$U^i := \bigoplus_{j=1}^s \mathbb{C} e^{\alpha_i + \beta_j} \otimes v_{\bar{\chi}}, \quad 1 \leq i \leq r,$$

forms an irreducible $\mathbb{C}[C_W]$-module. Set $W^i := V(\alpha_i) \boxtimes V(0) W$ for $1 \leq i \leq r$. Let $I^{\alpha,i}(\cdot, z)$ be a $V(0)$-intertwining operator of type $V(\alpha) \times W^i \rightarrow V(\alpha) \boxtimes V(0) W^i$. Since all $V(\alpha)$, $\alpha \in C$, are simple current $V(0)$-modules, $I^{\alpha,i}(\cdot, z)$ are unique up to scalars. It is possible to choose these intertwining operators to satisfy

$$(z_0 + z_2)^m I^{\alpha,j'}(x^\alpha, z_0 + z_2) I^{\beta,j}(x^\beta, z_2) w^j = \varepsilon(\alpha, \beta)(z_2 + z_0)^m I^{\alpha + \beta,j}(Y_{\bar{\chi}}(x^\alpha, z_0) x^\beta, z_2) w^j$$

for $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$, $w^j \in W^j$, $\alpha_j + C_W = \beta + \alpha_j + C_W$ and $m \gg 0$ (cf. [M2, Y2]). We can also choose $I^{0,j}(\cdot, z)$ so that $I^{0,j}(1, z) = \text{id}_{W^i}$. Now put

$$M = \text{Ind}_{V_H}(W, \bar{\chi}) := \bigoplus_{i=1}^r W^i \otimes U^i$$

and define a vertex operator map $Y(\cdot, z) : V_C \times M \rightarrow M((z))$ by

$$Y(x^\alpha, z) w^i \otimes u^i := I^{\alpha,i}(x^\alpha, z) w^i \otimes (e^\alpha \cdot u^i)$$

for $x^\alpha \in V^\alpha$, $w^i \in W^i$ and $u^i \in U^i$.

**Theorem 3.12.** ([M3, L3, Y2]) The induced module $\text{Ind}_{V_H}(W, \bar{\chi})$ equipped with the vertex operator map defined above is an irreducible $V_C$-module. Moreover, every irreducible $V_C$-module is obtained by an induced module.

**Remark 3.13.** Even if $\tau(W) \not\subset C^\perp$, one can still define an irreducible $\mathbb{Z}_2$-twisted $V_C$-module structure on $\text{Ind}_{V_H}(W, \bar{\chi})$ (cf. [L1, Y1]).

**Parameterization by a pair of binary codewords.** The irreducible $V_C$-modules can also be parameterized by a pair of binary codewords. For given $\beta \in C^\perp$ and $\gamma \in \mathbb{Z}_2^s$, we define a weight vector $h_{\beta, \gamma} = (h_{\beta, \gamma}^1, \ldots, h_{\beta, \gamma}^n)$, $h_{\beta, \gamma}^i \in \{0, 1/2, 1/16\}$ by

$$h_{\beta, \gamma}^i := \begin{cases} 
1/16 & \text{if } \beta_i = 1, \\
\gamma_i/2 & \text{if } \beta_i = 0.
\end{cases}$$

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Let $L(h_{\beta,\gamma}) = L(h_{\beta,\gamma}^1, \ldots, h_{\beta,\gamma}^n)$ be the irreducible $L(\frac{1}{2}, 0)^{\otimes n}$-module with the weight $h_{\beta,\gamma}$. Set $C_\beta := \{\alpha \in C \mid \text{supp}(\alpha) \subset \text{supp}(\beta)\}$ and take a maximal self-orthogonal subcode $H$ of $C_\beta$. Define a character $\chi_\beta$ of $H$ by $\chi_\beta(\alpha) := (-1)^{\langle \gamma, \alpha \rangle}$ for $\alpha \in H$ and extend to a character $\tilde{\chi}_\beta$ of $\pi_{C_\beta}^{-1}(H)$ by $\tilde{\chi}_\beta(\alpha, u) := \chi_\beta(\alpha)u$ for $\alpha \in H$ and $u \in \mathbb{C}^*$. Note that $\tilde{\chi}_\beta$ also defines a linear character on $\tilde{H}$. Then by Theorem 3.12, the pair $(\beta, \gamma)$ determines an irreducible $V_C$-module

$$M_{C}(\beta, \gamma) := \text{Ind}_{V_H}^{V_C}(L(h_{\beta,\gamma}), \tilde{\chi}_\gamma).$$

Note that if $C$ is self-orthogonal and $\text{supp}(C) \subset \text{supp}(\beta)$, then $M_{C}(\beta, \gamma) \simeq L(h_{\beta,\gamma})$ as a $V(0)$-module. It is also obvious that

$$M_{C}(0, \gamma) = \bigoplus_{\alpha=(\alpha_1, \ldots, \alpha_n) \in C+\gamma} L(\frac{1}{2}, \alpha_1/2) \otimes \cdots \otimes L(\frac{1}{2}, \alpha_n/2).$$

This module is called a coset module in $\text{M3}^2$. We sometimes denote $M_{C}(0, \gamma)$ simply by $V_{C+\gamma}$.

We shall review some basic properties of $M_{C}(\beta, \gamma)$.

**Lemma 3.14. (DGL)** The definition of $M_{C}(\beta, \gamma)$ is independent of the choice of the maximal self-orthogonal subcode $H$ of $C_\beta$ and the section map $\iota : H \to \mathbb{C}^*$ which realizes an embedding $H \to \pi_{C_\beta}^{-1}(H)$ in (3.12).

**Proof:** Let $H'$ be a maximal self-orthogonal subcode of $C_\beta$. Then the preimage $\pi_{C_\beta}(H')$ of $H'$ in (3.12) splits so that one can take a section map $\iota' : H' \to \mathbb{C}^*$ such that the mapping $H' \ni \alpha \mapsto (\alpha, \iota'(\alpha)) \in \tilde{H}'$ is a group homomorphism. Define a linear character $\chi_{\beta}'$ of $\pi_{C_\beta}^{-1}(H')$ by $\tilde{\chi}_{\beta}'(\alpha, \iota'(\alpha)u) := (-1)^{\langle \gamma, \alpha \rangle}u$ for $\alpha \in H'$ and $u \in \mathbb{C}^*$. Then $\tilde{\chi}_{\beta}'$ is also a linear character of the preimage $\tilde{H}'$ of $H'$ in (3.11). We shall show that $\text{Ind}_{V_H}^{V_C}(L(h_{\beta,\gamma}), \tilde{\chi}_\gamma) \simeq \text{Ind}_{V_H}^{V_C}(L(h_{\beta,\gamma}), \tilde{\chi}_{\beta}')$ for this. It suffices to show that $\text{Res}_{\tilde{H}'}^{\tilde{H}} \text{Ind}_{H}^{\tilde{H}} \tilde{\chi}_{\gamma}$ contains $\tilde{\chi}_{\beta}'$, by Theorem 3.11 and the construction of induced modules. Let $C_{\beta}[\mathbb{C}^*] = \bigoplus_{n \in \mathbb{Z}^n} \mathbb{C}^n \otimes \mathbb{C}^n$ be the twisted group algebra of $C_{\beta}$ associated to 2-cocycle $\varepsilon$ and let $\mathbb{C}v_{\tilde{\chi}_{\gamma}}$ be a $\mathbb{C}^n[H]$-module such that $\iota(\alpha)e^\alpha \cdot v_{\tilde{\chi}_{\gamma}} = (-1)^{\langle \gamma, \alpha \rangle}v_{\tilde{\chi}_{\gamma}}$. Then $\text{Ind}_{H}^{\tilde{H}} \tilde{\chi}_{\gamma} = \mathbb{C}^n[C_{\beta}] \otimes_{\mathbb{C}^n[H]} v_{\tilde{\chi}_{\gamma}}$. Set

$$x := \sum_{\delta \in H'} (-1)^{\langle \gamma, \delta \rangle} \iota'(\delta)e^\delta \otimes v_{\tilde{\chi}_{\gamma}} \in \text{Ind}_{H}^{\tilde{H}} \tilde{\chi}_{\gamma}.$$ 

It is easy to see that $x \neq 0$ and $\iota'(\alpha)e^\alpha \cdot x = (-1)^{\langle \gamma, \alpha \rangle}x$ for all $\alpha \in H'$ so that $\mathbb{C}x$ is a $\tilde{H}'$-submodule affording a character $\tilde{\chi}_{\beta}'$. This completes the proof.

Similarly, one can show the following by considering linear characters of $\tilde{H}$.

**Lemma 3.15. (DGL)** Let $\beta, \beta' \in C^+$ and $\gamma, \gamma' \in \mathbb{Z}_2^n$. Then the irreducible $V_C$-modules $M_{C}(\beta, \gamma)$ and $M_{C}(\beta', \gamma')$ are isomorphic if and only if

$$\beta = \beta' \quad \text{and} \quad \gamma + \gamma' \in C + H^{\perp_{\beta}},$$

where $H$ is a maximal self-orthogonal subcode of $C_{\beta}$ and $H^{\perp_{\beta}} = \{\alpha \in \mathbb{Z}_2^n \mid \text{supp}(\alpha) \subset \text{supp}(\beta) \text{ and } \langle \alpha, \delta \rangle = 0 \text{ for all } \delta \in H\}$.

---

3 This name has nothing to do with so-called the coset construction of a commutant subalgebra.
In particular, every irreducible orthogonal subcode of $C$ Let Corollary 3.16.

Proof: Assume that $M_C(\beta, \gamma) \simeq M_C(\beta', \gamma')$. Then clearly $\beta = \beta'$ by 1/16-word decompositions. It is also obvious from the definition of $M_C(\beta, \gamma)$ that $M_C(\beta, \gamma) \simeq M_C(\beta, \gamma + \delta)$ for any $\delta \in H^{1,\beta}$. Take a maximal self-orthogonal subcode $H$ of $C_{\beta}$ and a section map $\iota : H \to \{\pm 1\}$. Let $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\delta_1, \ldots, \delta_s\}$ be transversals of $C/H$ and $C_{\beta}/H$, respectively. Then by definition we have a decomposition

$$M_C(\beta, \gamma) = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \left( V(\alpha_i) \otimes L(h_{\beta, \gamma}) \right) \otimes_{\mathbb{C}} \left( e^{\alpha_i + \delta_j} \otimes \bar{\chi}_\gamma \right).$$

It follows from (3.14) that

$$\left( V^{\alpha_i}_{\mathbb{C}} \otimes L(h_{\beta, \gamma}) \right) \otimes_{\mathbb{C}} \left( e^{\alpha_i + \delta_j} \otimes \bar{\chi}_\gamma \right) \simeq M_H(\beta, \gamma + \alpha_i + \delta_j)$$

as a $V_H$-submodule. Therefore, we have the following decompositions:

$$M_C(\beta, \gamma) = \bigoplus_{\delta + H \in C/H} M_H(\beta, \gamma + \delta), \quad M_C(\beta, \gamma') = \bigoplus_{\delta + H \in C/H} M_H(\beta, \gamma' + \delta).$$

Since $H = C_{\beta} \cap H^{1,\beta}$ by the maximality of $H$, all $M_C(\beta, \gamma + \delta), \delta \in C/H$, are inequivalent irreducible $V_H$-submodules. Thus, if $M_C(\beta, \gamma) \simeq M_C(\beta, \gamma')$, then $\bar{\chi}_{\gamma'} = \bar{\chi}_{\gamma + \delta}$ for some $\delta \in C$. This is possible if and only if $\gamma + \gamma' \in C + H^{1,\beta}$. Conversely, if $\gamma + \gamma' \in C + H^{1,\beta}$, then $M_C(\beta, \gamma)$ and $M_C(\beta, \gamma')$ contain isomorphic irreducible $V_H$-submodules. Since $V_C$-module structures on $M_C(\beta, \gamma)$ and $M_C(\beta, \gamma')$ are uniquely determined by their $V_H$-module structures, they are isomorphic.

In the proof above, we have shown

**Corollary 3.16.** Let $M_C(\beta, \gamma)$ be an irreducible $V_C$-module. Let $H$ be a maximal self-orthogonal subcode of $C_{\beta}$. Then as a $V_H$-module,

$$M_C(\beta, \gamma) = \bigoplus_{\delta + H \in C/H} M_H(\beta, \gamma + \delta).$$

In particular, every irreducible $V_H$-submodule of $M_C(\beta, \gamma)$ is multiplicity-free.

### 3.3.3 Dual module

We shall determine the structure of the dual module of a $V_C$-module $M_C(\beta, \gamma)$. Recall that the dual module of a $V$-module $M = \oplus_{n \in \mathbb{N}} M_{n+h}$ is defined to be its restricted dual $M^* = \oplus_{n \in \mathbb{N}} M_{n+h}^*$ equipped with a vertex operator $Y^*_M(\cdot, z)$ defined by (3.5).

First, we consider the case when the code is self-orthogonal. Let $H$ be a self-orthogonal code of $\mathbb{Z}_2^n$. In this case, one can define a character $\varphi$ of $H$ by $\varphi(\alpha) = (-1)^{\text{wt}(\alpha)/2}$ for $\alpha \in H$. So there exists a codeword $\kappa \in \mathbb{Z}_2^n$ such that $\varphi(\alpha) = (-1)^{(\kappa, \alpha)}$ for all $\alpha \in H$.

**Lemma 3.17.** Let $H < \mathbb{Z}_2^n$ be a self-orthogonal code. For any $\gamma \in \mathbb{Z}_2^n$, the dual module of $M_H((1^n), \gamma)$ is isomorphic to $M_H((1^n), \gamma + \kappa)$, where $\kappa \in \mathbb{Z}_2^n$ is such that $(-1)^{(\kappa, \alpha)} = (-1)^{\text{wt}(\alpha)/2}$ for all $\alpha \in H$.  

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Proof: By assumption, $M_H((1^n), \gamma) = L(\frac{1}{2}, \frac{1}{16})^\otimes n \otimes \bar{\chi}_{\gamma}$ and $M_H((1^n), \gamma)$ is an irreducible $V(0)$-module. Therefore, $M_H((1^n), \gamma)^* \simeq M_H((1^n), \gamma')$ for some $\gamma' \in \mathbb{Z}_2^n$. Since $L(\frac{1}{2}, \frac{1}{16})^\otimes n$ is a self-dual $V(0)$-module, we can take a $V(0)$-isomorphism $f : M \to M^*$. Let $Y(\cdot, z)$, $Y^*(\cdot, z)$ be the vertex operator map on $M_H((1^n), \gamma)$, $M_H((1^n), \gamma)^*$, respectively. For $\alpha \in H$, let $x^\alpha \in V(\alpha)$ be a non-zero highest weight vector of weight $\text{wt}(\alpha)/2$. Then $Y^*(x^\alpha, z)f$ and $fY(x^\alpha, z)$ are $V(0)$-intertwining operators of type

$$V(\alpha) \times M_H((1^n), \gamma) \to M_H((1^n), \gamma)^*.$$ 

Since the space of $V(0)$-intertwining operators of this type is one-dimensional, there exists a scalar $\lambda_\alpha \in \mathbb{C}^*$ such that $Y^*(x^\alpha, z)f = \lambda_\alpha fY(x^\alpha, z)$. Let $v$ be a non-zero highest weight vector of $M_H((1^n), \gamma)$. Then by (3.5), one has

$$\langle Y^*(x^\alpha, z)f, v \rangle = (-1)^{\text{wt}(\alpha)/2} z^{-\text{wt}(\alpha)} \langle f, Y(x^\alpha, z^{-1})v \rangle = (-1)^{\text{wt}(\alpha)/2} z^{-\text{wt}(\alpha)/2} \langle f, x^\alpha_{(\text{wt}(\alpha)/2-1)}v \rangle.$$ 

On the other hand,

$$\langle Y^*(x^\alpha, z)f, v \rangle = \lambda_\alpha \langle f, Y(x^\alpha, z)v, v \rangle = \lambda_\alpha z^{-\text{wt}(\alpha)/2} \langle f, x^\alpha_{(\text{wt}(\alpha)/2-1)}v, v \rangle.$$ 

Since $x^\alpha_{(\text{wt}(\alpha)/2-1)}v = tv$ for some $t \in \mathbb{C}^*$ and $\langle f, v, v \rangle \neq 0$, we have $\lambda_\alpha = (-1)^{\text{wt}(\alpha)/2} = (-1)^{(\kappa, \alpha)}$. Therefore, by considering the linear character associated to $M_H((1^n), \gamma)^*$, we see that $M_H((1^n), \gamma)^* \simeq M_H((1^n), \gamma + \kappa).$ \hfill \rule{2mm}{2mm}

Proposition 3.18. Let $C$ be an even linear code, $\beta \in C^\perp$ and $\gamma \in \mathbb{Z}_2^n$. Let $H$ be a maximal self-orthogonal subcode of $C_\beta$. Then the dual module $M_C(\beta, \gamma)^*$ is isomorphic to $M_C(\beta, \gamma + \kappa_H)$ where $\kappa_H \in \mathbb{Z}_2^n$ is such that $\text{supp}(\kappa_H) \subset \text{supp}(\beta)$ and $(-1)^{(\kappa_H, \alpha)} = (-1)^{\text{wt}(\alpha)/2}$ for all $\alpha \in H$.

Proof: By Corollary 3.16, $M_C(\beta, \gamma)$ contains a $V_H$-submodule

$$M_H(\beta, \gamma) \simeq L(h_\beta, \gamma) \otimes \bar{\chi}_\gamma.$$ 

By the previous lemma, the dual module $M_C(\beta, \gamma)^*$ contains a $V_H$-submodule isomorphic to

$$M_H(\beta, \gamma)^* \simeq M_H(\beta, \gamma + \kappa_H) = L(h_\beta, \gamma) \otimes \bar{\chi}_{\gamma + \kappa_H}.$$ 

Therefore, by the structure of irreducible $V_C$-modules, $M_C(\beta, \gamma)^* = \text{Ind}_{V_H}^{V_C}(L(h_\beta, \gamma), \chi_{\gamma + \kappa_H})$ and hence $M_C(\beta, \gamma)^* \simeq M_C(\beta, \gamma + \kappa_H)$. \hfill \rule{2mm}{2mm}

As an immediate corollary, we have:

Corollary 3.19. With reference to the proposition above, $M_C(\beta, \gamma)$ is self-dual if and only if $\kappa_H \in C$. In particular, $M_C(0, \gamma)$ is self-dual for all $\gamma \in \mathbb{Z}_2^n$. 

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3.3.4 Fusion rules

We shall compute the fusion rules among some irreducible $V_C$-modules. First, we recall a result from [M2].

**Lemma 3.20.** ([M2]) For $\alpha, \beta \in \mathbb{Z}_2^n$, $M_C(0, \alpha) \boxtimes V C M_C(0, \beta) = M_C(0, \alpha + \beta)$.

**Proof:** Follows from Proposition 3.5.

By the lemma above, we see that $M_C(0, \alpha) \boxtimes V C M_C(0, \alpha) = M_C(0, 0) = V_C$. Therefore, all $M_C(0, \alpha)$, $\alpha \in \mathbb{Z}_2^n$, are simple current $V_C$-modules by Corollary 3.2. It also follows that $M_C(0, \alpha) \boxtimes V C M_C(\beta, \gamma)$ is an irreducible $V_C$-module with the $1/16$-word $\beta$. The corresponding fusion rules are also computed by Miyamoto [M3].

**Lemma 3.21.** ([M3]) Let $\alpha, \beta, \gamma \in \mathbb{Z}_2^n$ with $\beta \in C^\perp$. Then

$$M_C(0, \alpha) \boxtimes V C M_C(\beta, \gamma) = M_C(\beta, \alpha + \gamma).$$

Moreover, the top weight of $M_C(\beta, \gamma)$ and the top weight of $M_C(\beta, \alpha + \gamma)$ are congruent to $(\alpha, \beta)/2$ modulo $\mathbb{Z}$.

**Proof:** The assertion is proved in Lemma 3.27 of [M3] in the case $\text{supp}(\alpha) \subset \text{supp}(\beta)$. We generalize his idea to obtain the desired fusion rule. Since $M_C(0, \alpha)$ is a simple current $V_C$-module, we know that there exists $\gamma \in \mathbb{Z}_2^n$ such that $M_C(0, \alpha) \boxtimes V C M_C(\beta, \gamma) = M_C(\beta, \gamma')$. Therefore, if we can construct a non-zero $V_C$-intertwining operator of type $M_C(0, \alpha) \times M_C(\beta, \gamma) \rightarrow M_C(\beta, \alpha + \gamma)$, then we are done. To do this, we have to extend $V_C$ to a larger algebra. The case $\alpha \in C$ is trivial so that we assume that $\alpha \notin C$. Set $C' = C \sqcup (C + \alpha)$. We can define a simple vertex operator (super)algebra structure on the space $V_C = \oplus V_{C + \alpha} = M_C(0, 0) \oplus M_C(0, \alpha)$. This is a VOA if $\alpha$ is even, and an SVOA if $\alpha$ is odd. Let $H'$ be a subcode of $C'$ such that $H'$ is a maximal abelian subgroup of $C'$ in the extension $[3.11]$. In the definition of the induced module $M_C(\beta, \gamma) = \text{Ind}^V C_{\beta, \gamma}(h(\beta, \gamma), \tilde{\chi}_\gamma)$, if we use $C'|C'$ instead of $C'|C$ then we obtain an irreducible $V_{C'}$-module

$$M_{C'}(\beta, \gamma) := \text{Ind}^V C_{\beta, \gamma}(h(\beta, \gamma), \tilde{\chi}_\gamma))$$

which contains $M_C(\beta, \gamma)$ as a $V_C$-submodule. The structure $M_{C'}(\beta, \gamma)$ is an untwisted $V_{C'}$-module if $\langle \alpha, \beta \rangle = 0$ and otherwise it is a $\mathbb{Z}_2$-twisted $V_{C'}$-module (cf. [L1, Y2]). Nevertheless, the subspace $M = V_{C + \alpha} \cdot M_C(\beta, \gamma)$ of $M_{C'}(\beta, \gamma)$ is an irreducible $V_C$-submodule. Let $H$ be a maximal self-orthogonal subcode of $C'$ (not $C'$). It is easy to see that there exists an irreducible $V_H$-submodule of $M$ isomorphic to $M_H(\beta, \alpha + \gamma)$. Then $M \simeq M_C(\beta, \alpha + \gamma)$ by the structure of an irreducible $V_C$-module. Since $M \simeq M_C(0, \alpha) \boxtimes V C M_C(\beta, \gamma)$, we obtain the desired fusion rule.

Since the powers of $z$ in $L(1/2, 0)$-intertwining operators of type $L(1/2, h/2) \times L(1/2, 1/16) \rightarrow L(1/2, 1/16)$ are in $h/2 + \mathbb{Z}$ for $h \in \{0, 1\}$, the difference of top weights is as in the assertion.

By this lemma, we can compute the following fusion rule.
Proposition 3.22. Let $\beta \in C^\perp$ and $\gamma \in \mathbb{Z}_2^n$. Let $H$ be a maximal self-orthogonal subcode of $C_\beta$. Then

$$M_C(\beta, \gamma) \boxtimes_{V_C} M_C(\beta, \gamma)^* = \sum_{\delta \in C + H^{\perp, \beta}} M_C(0, \delta),$$

where in the display above $\delta \in \mathbb{Z}_2^n$ runs over a transversal of $(C + H^{\perp, \beta})/C$.

Proof: By the 1/16-word decomposition, $M_C(\beta, \gamma) \boxtimes_{V_C} M_C(\beta, \gamma)^*$ contains only modules of type $M_C(0, \delta)$. Now assume that $(M_C(\beta, \gamma)^* M_C(0, \delta))_{V_C} \neq 0$. Then by the symmetry of fusion rules, we have

$$\left( \begin{array}{c} M_C(0, \delta) \\ M_C(\beta, \gamma) \\ M_C(\beta, \gamma)^* \end{array} \right)_{V_C} \approx \left( \begin{array}{c} M_C(\beta, \gamma) \\ M_C(\beta, \gamma) \\ M_C(0, \delta) \end{array} \right)_{V_C} \neq 0.$$

Since $M_C(\beta, \gamma) \boxtimes_{V_C} M_C(0, \delta) = M_C(\beta, \gamma + \delta)$ by the previous lemma, this is possible if and only if $\delta \in C + H^{\perp, \beta}$ by Lemma 3.15. Therefore, we have the fusion rule as stated. □

By the lemma above, we introduce the following definition.

Definition 3.23. Let $\beta \in \mathbb{Z}_2^n$ and $H$ a subcode with $\text{supp}(H) \subset \text{supp}(\beta)$. $H$ is said to be self-dual with respect to (w.r.t.) $\beta$ if $H = H^{\perp, \beta}$.

Remark 3.24. Note that if $H$ is a self-dual subcode of $C_\beta$ w.r.t. $\beta$ then $C + H^{\perp, \beta} = C$.

Corollary 3.25. $M_C(\beta, \gamma)$ is a simple current module if and only if $C_\beta$ contains a self-dual subcode w.r.t. $\beta$.

Proof: Follows from Corollary 3.22 and Proposition 3.22. □

Remark 3.26. Now suppose $M_C(\beta, 0) \boxtimes_{V_C} M_C(\beta, 0)^* = \sum_{i=1}^p M_C(0, \delta_i)$. Let $H$ be a maximal self-orthogonal subcode of $C_\beta$ we used to define $M_C(\beta, \gamma)$, and let $\kappa_H \in (\mathbb{Z}_2^n)_\beta$ such that $\langle \kappa_H, \alpha \rangle = \langle \alpha, \alpha \rangle/2$ mod 2 for all $\alpha \in H$ as in Proposition 3.18. Then $M_C(\beta, 0)^* = M_C(\beta, \kappa_H) = M_C(0, \kappa_H) \boxtimes_{V_C} M_C(\beta, 0)$ and thus $M_C(\beta, 0) = M_C(0, \kappa_H) \boxtimes_{V_C} M_C(\beta, 0)^*$. Using this, we can compute the following fusion rule:

$$M_C(\beta, \gamma_1) \boxtimes_{V_C} M_C(\beta, \gamma_2) = \left\{ M_C(0, \gamma_1) \boxtimes_{V_C} M_C(\beta, 0) \right\} \boxtimes_{V_C} \left\{ M_C(0, \gamma_2) \boxtimes_{V_C} M_C(\beta, 0) \right\}$$

$$= M_C(0, \gamma_1 + \gamma_2) \boxtimes_{V_C} M_C(\beta, 0) \boxtimes_{V_C} M_C(\beta, 0)$$

$$= M_C(0, \gamma_1 + \gamma_2) \boxtimes_{V_C} M_C(\beta, 0) \boxtimes_{V_C} \left\{ M_C(0, \kappa_H) \boxtimes_{V_C} M_C(\beta, 0)^* \right\}$$

$$= M_C(0, \gamma_1 + \gamma_2 + \kappa_H) \boxtimes_{V_C} \left\{ \sum_{i=1}^p M_C(0, \delta_i) \right\}$$

$$= \sum_{i=1}^p M_C(0, \gamma_1 + \gamma_2 + \kappa_H + \delta_i).$$

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3.4 Structure of framed VOAs

We shall prove that every framed VOA is a simple current extension of a code VOA. This result has many fruitful consequences. For example, the irreducible representations of a framed VOA can be determined by a notion of induced modules. Another interesting result is the conditions on possible structure codes of holomorphic framed VOAs.

3.4.1 Simple current structure

In this subsection we discuss how a code VOA can be extended to a framed VOA. First, we give a construction of a non-trivial simple current extension.

Lemma 3.27. Let $\beta \in C^\perp$ be a non-zero codeword. Let $M_C(\beta, \gamma)$ be an irreducible $V_C$-module with an integral top weight. If $C_\beta$ contains a doubly even self-dual subcode w.r.t. $\beta$, then there exists a unique structure of a framed VOA on $V_C \oplus M_C(\beta, \gamma)$ which forms a $\mathbb{Z}_2$-graded simple current extension of $V_C$.

Proof: Let $H$ be a doubly even self-dual subcode of $C_\beta$ w.r.t. $\beta$. By Proposition 3.18, $M_C(\beta, \gamma)$ is self-dual, and by Corollary 3.25, $M_C(\beta, \gamma)$ is a simple current $V_C$-module. By Corollary 3.16, $M_C(\beta, \gamma)$ has a $V_H$-module structure

$$M_C(\beta, \gamma) = \bigoplus_{\delta + H \in C/H} M_H(\beta, \gamma + \delta).$$

It is clear that a $V_C$-invariant bilinear form on $M_C(\beta, \gamma)$ induces a $V_H$-invariant bilinear form on $M_H(\beta, \gamma)$. It is shown in [Li1] that a $V_C$-invariant bilinear form on $M_C(\beta, \gamma)$ is either symmetric or skew-symmetric. Since the top level of $M_H(\beta, \gamma + \delta)$ is one-dimensional, the $V_C$-invariant bilinear form on $M_C(\beta, \gamma)$ must be symmetric. Therefore, $V_C \oplus M_C(\beta, \gamma)$ forms a $\mathbb{Z}_2$-graded simple current extension of $V_C$ by Proposition 3.6.

Lemma 3.28. Let $\beta \in C^\perp$ and $\gamma \in \mathbb{Z}_2^n$ with $\beta \neq 0$. Assume that $V = V_C \oplus M_C(\beta, \gamma)$ is a framed VOA. Then $C_\beta$ contains a maximal self-orthogonal and doubly even subcode $K$.

Proof: Let $H$ be a maximal self-orthogonal subcode of $C_\beta$. If $H$ is doubly even, then we are done. So we assume that $H$ contains a codeword whose weight is congruent to 2 modulo 4. Since $V = V_C \oplus M_C(\beta, \gamma)$ forms a simple VOA, $M_C(\beta, \gamma)$ is a self-dual $V_C$-module. Therefore, by Corollary 3.19, there exists a codeword $\kappa_H \in C_\beta$ such that $(-1)^{[\kappa_H, \alpha]} = (-1)^{wt(\alpha)/2}$ for all $\alpha \in H$. By Corollary 3.16, $M_C(\beta, \gamma)$ has the following decomposition as a $V_H$-module:

$$M_C(\beta, \gamma) = \bigoplus_{\delta + H \in C/H} M_H(\beta, \gamma + \delta).$$

By our choice of $H$, $\kappa_H \not\in H^{\perp \beta}$ so that $M_H(\beta, \gamma)$ and its dual $M_H(\beta, \gamma + \kappa_H)$ are inequivalent irreducible $V_H$-submodules of $V$. We consider a sub VOA $U$ generated by $M_H(\beta, \gamma) \oplus M_H(\beta, \gamma + \kappa_H)$. By the fusion rule given in Proposition 3.22, $U$ has the following shape as a $V_H$-module:

$$U = M_H(0, 0) \oplus M_H(0, \kappa_H) \oplus M_H(\beta, \gamma) \oplus M_H(\beta, \gamma + \kappa_H).$$

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Note that $H = C \cap H^{+\beta}$ by the maximality of $H$. Set $H' := H \cup (H + \kappa H)$ and $H_0 := H \cap \langle \kappa H \rangle^\perp$, and then take any $\alpha' \in H \setminus H_0$. Then $H' = H_0 \cup (H_0 + \alpha') \cup (H_0 + \kappa H) \cup (H_0 + \alpha' + \kappa H)$. We set $K := H_0 \cup (H_0 + \kappa H)$. It is clear that $U$ also possesses a symmetric invariant bilinear form which we shall denote by $\langle \cdot , \cdot \rangle_U$. Since $M_H(\beta, \gamma)$ and $M_H(\beta, \gamma + \kappa H)$ are dual to each other, we have

$$\langle M_H(\beta, \gamma), M_H(\beta, \gamma) \rangle_U = \langle M_H(\beta, \gamma + \kappa H), M_H(\beta, \gamma + \kappa H) \rangle_U = 0.$$ 

By Lemma 3.15, $M_H(\beta, \gamma)$ and $M_H(\beta, \gamma + \kappa H)$ are isomorphic irreducible $V_{H_0}$-modules. Therefore, there exists a $V_{H_0}$-isomorphism $\varphi : M_H(\beta, \gamma) \rightarrow M_H(\beta, \gamma + \kappa H)$. Let $u$ be a non-zero highest weight vector of $M_H(\beta, \gamma)$. Since the top level of $M_H(\beta, \gamma)$ is one-dimensional, we may assume that $\langle u, \varphi(u) \rangle_U = 1$. Now consider the decomposition (3.15) of $U$ with respect to a series of sub VOAs $V_{H_0} \subset V_K \subset V_{H'}$ of $U$. It is clear that $M_H(0, 0) \oplus M_H(0, \kappa H) = M_K(0, 0) \oplus M_K(0, \alpha')$. Therefore, there exists a decomposition of $U$ as a $V_K$-module

$$U = M_K(0, 0) \oplus M_K(0, \alpha') \oplus W$$

with $W = M_H(\beta, \gamma) \oplus M_H(\beta, \gamma + \kappa H)$. Let $W^0$ be an irreducible $V_K$-submodule of $W$. Since $K$ is a self-orthogonal submodule of $C_\beta$, the top level of $W^0$ is one-dimensional. Let $v \in W^0$ be a non-zero highest weight vector. As we mentioned, $M_H(\beta, \gamma)$ and $M_H(\beta, \gamma + \kappa H)$ are isomorphic $V_{H_0}$-submodules. But $M_H(\beta, \gamma)$ and $M_H(\beta, \gamma + \kappa H)$ are not $V_K$-submodules by the fusion rule of $V_{H_0}$-modules. Therefore, we can write $v = c_1 u + c_2 \varphi(u)$ with $c_1, c_2 \neq 0$. This shows that $\langle v, v \rangle_U = 2c_1 c_2 \neq 0$ so that $W^0$ is a self-dual $V_K$-submodule. Then $K$ is a doubly even code by Corollary 3.19. Since $|K| = |H| = 2|H_0|$, $K$ is a maximal self-orthogonal submodule of $C_\beta$. Therefore, $C_\beta$ contains the desired subcode $K$.

We recall the following fact from the coding theory.

**Theorem 3.29.** (McST) Let $n$ be divisible 8 and $H$ a doubly even code of $\mathbb{Z}_2^n$ containing all one vector $(1 \ldots 1) \in \mathbb{Z}_2^n$. Then there exists a doubly even self-dual code $H'$ such that $H \subset H'$.

Now we begin to prove that every framed VOA is a simple current extension of a code VOA. For this, it suffices to show the following proposition.

**Proposition 3.30.** Let $\beta \in C^{\perp}$ and $\gamma \in \mathbb{Z}_2^n$. Assume that $V = V_C \oplus M_C(\beta, \gamma)$ forms a framed VOA. Then $C_\beta$ contains a doubly even self-dual subcode w.r.t. $\beta$.

**Proof:** By Lemma 3.28, $C_\beta$ contains a maximal self-orthogonal subcode $H$ which is doubly even. By Corollary 3.16, $V$ has a decomposition

$$V = \bigoplus_{\delta + H \in C/H} M_H(0, \delta) \oplus M_H(\beta, \gamma + \delta)$$

as a $V_H$-module. By the fusion rule in Proposition 3.22, the subspace

$$U := V_H \oplus M_H(\beta, \gamma)$$

forms a sub VOA of $V$, since $H = C_\beta \cap H^{\perp\beta}$. If $H$ is not self-dual, then there exists a doubly even self-dual subcode $H'$ of $\mathbb{Z}_2^n$ w.r.t. $\beta$ such that $H \cup (H + \beta) \subset H'$ by Theorem 3.29.
Note that the weight of $\beta$ is divisible by 8 since $M_C(\beta, \gamma)$ has an integral top weight. Let us consider the code VOA $V_{H'}$ associated to $H'$. It is clear that $V_{H'}$ contains $V_H$ as a sub VOA. By the structure theory in Theorem 3.12, we can define an irreducible $V_{H'}$-module $M_{H'}(\beta, \gamma)$ such that $M_{H'}(\beta, \gamma)|_{V_H} \cong M_H(\beta, \gamma)$ as a $V_H$-module. For simplicity, we shall denote $M_{H'}(\beta, \gamma)$ by $W$. Since the top level of $W$ is one-dimensional, the $V_{H'}$-invariant bilinear form on $W$ is symmetric. Therefore, by Proposition 3.5, we can define a framed VOA structure on $U' := V_{H'} \oplus W$.

We denote the vertex operator map on $U'$ by $Y'(\cdot, z)$. Now suppose $H$ is a proper subcode of $H'$. Then

$$V_{H'} = \bigoplus_{\delta + H \in H'/H} V_{\delta + H}, \quad V_{\delta + H} = M_H(0, \delta),$$

as a $V_H$-module. Let $\pi_{\delta + H} : V_{H'} \to V_{\delta + H}$ be the projection map. Then for $u, v \in W$, we have

$$Y'(u, z)v = \sum_{\delta + H \in H'/H} \pi_{\delta + H} Y'(u, z)v.$$ 

Since the simple VOA structure is unique on $V_{H'} \oplus W$, we may assume that $\pi_H Y'(u, z)v = Y_V(u, z)v$. Take any $\alpha \in H' \setminus H$ and set $K := H \sqcup (H + \alpha)$. We shall show the following claim:

**Claim.** For $u, v \in W$, there exists $N = N(u, v) \in \mathbb{N}$ such that

$$(z_1 - z_2)^N Y'(u, z_1) \pi_{H + \alpha} Y'(v, z_2)w = (z_1 - z_2)^N Y'(v, z_2) \pi_{H + \alpha} Y'(u, z_1)w \quad (3.16)$$

for any $w \in W$.

For, take any $a \in V_{H + \alpha}$. Since $U = V_H \oplus M_H(\beta, \gamma)$ forms a framed VOA by assumption, there exists $N = N(u, v) \in \mathbb{N}$ such that

$$(z_1 - z_2)^N Y'(u, z_1) \pi_H Y'(v, z_2)w = (z_1 - z_2)^N Y'(v, z_2) \pi_H Y'(u, z_1)w. \quad (3.17)$$

Take a sufficiently large $k \in \mathbb{N}$. Then one has

$$(z_1 - z_2)^N (z_0 - z_1)^k (z_0 - z_2)^k Y'(u, z_1) \pi_{H + \alpha} Y'(v, z_2)Y'(a, z_0)w$$

$$= (z_1 - z_2)^N (z_0 - z_1)^k (z_0 - z_2)^k Y'(a, z_0) Y'(u, z_1) \pi_H Y'(v, z_2)w$$

$$= (z_1 - z_2)^N (z_0 - z_1)^k (z_0 - z_2)^k Y'(a, z_0) Y'(v, z_2) \pi_H Y'(u, z_1)w$$

$$= (z_1 - z_2)^N (z_0 - z_1)^k (z_0 - z_2)^k Y'(v, z_2) \pi_{H + \alpha} Y'(u, z_1) Y'(a, z_0)w.$$

Since the expansions of both sides of the equations above have only finitely many negative powers of $z_0$, we get

$$(z_1 - z_2)^N Y'(u, z_1) \pi_{H + \alpha} Y'(v, z_2)w = (z_1 - z_2)^N Y'(v, z_2) \pi_{H + \alpha} Y'(u, z_1)w.$$ 

Note that $Y'(a, z) \pi_H = \pi_{H + \alpha} Y'(a, z)$ on $V_{H'}$ and $W = V_{H + \alpha} \cdot W$. 

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By the Claim above, we can introduce a framed VOA structure on $X := V_K \oplus M_K(\beta, \gamma)$ as follows. Since $W$ as a $V_K$-module is isomorphic to $M_K(\beta, \gamma)$, we can identify these structures. For $a, b \in V_K$ and $u, v \in M_K(\beta, \gamma)$, we define the vertex operator map $Y_X(\cdot, z)$ by

$$Y_X(a, z)b := Y'(a, z)b, \quad Y_X(a, z)u := Y'(a, z)u, \quad Y_X(u, z)a := Y'(u, z)a,$$

and

$$Y_X(u, z)v := \pi_H Y'(u, z)v + \pi_{H+a} Y'(u, z)v.$$ 

Let $\langle \cdot, \cdot \rangle_{U'}$ be a non-zero invariant bilinear form on $U'$, which is unique up to a scalar. Since $V_K$ is a subalgebra of $U'$, we can define an invariant bilinear form $\langle \cdot, \cdot \rangle_{V_K}$ on $V_K$ by $\langle a, b \rangle_{V_K} := \langle a, b \rangle_{U'}$. In addition, since $W$ as a $V_K$-module is isomorphic to $M_K(\beta, \gamma)$, we may view $\langle \cdot, \cdot \rangle_{U'}$ restricted on $W$ as a $V_K$-invariant bilinear form on $M_K(\beta, \gamma)$. Then

$$\langle a, Y_X(u, z)v \rangle_{V_K} = \langle a, \pi_H Y'(u, z)v \rangle_{V_K} + \langle a, \pi_{H+a} Y'(u, z)v \rangle_{V_K}$$

$$= \langle a, Y'(u, z)v \rangle_{U'} + \langle a, \pi_{H+a} Y'(u, z)v \rangle_{U'}$$

$$= \langle Y'(e^{zL(1)}(z^{-2})L(0)u, z^{-1})a, v \rangle_{U'}.$$ 

By the equality above, it follows from Section 5.6 of [FHL] and [Li2] that $Y_X(\cdot, z)$ satisfies the Jacobi identity if and only if we have a locality for any three elements in $M_K(\beta, \gamma)$, which follows from (3.16) and (3.17). Therefore, $(X, Y_X(\cdot, z))$ is also a framed VOA. In fact, one can define a framed VOA structure on $V_E \oplus M_E(\beta, \gamma)$ for any subcode $E$ of $H'$ containing $H$ in a similar way. We shall deduce a contradiction from this fact.

Let $W^1$ and $W^2$ be $V_H$-modules isomorphic to $M_H(\beta, \gamma)$. Since $M_K(\beta, \gamma) \simeq W^i$ as $V_H$-modules, we can take $V_H$-isomorphisms $\varphi_i : W^i \to M_K(\beta, \gamma) \subset X$ for $i = 1, 2$. Set

$$X' := V_H \oplus V_{H+a} \oplus W^1 \oplus W^2.$$ 

We shall define a vertex operator map $Y_{X'}$ on $X'$ as follows.

For $a^0, b^0 \in V_H$, $a^1, b^1 \in V_{H+a}$, $u^1, v^1 \in W^1$ and $u^2, v^2 \in W^2$, define

$$Y_{X'}(a^0, z) := \begin{bmatrix} Y_X(a^0, z) & 0 & 0 & 0 \\ 0 & Y_X(a^0, z) & 0 & 0 \\ 0 & 0 & \varphi_1^{-1}Y_X(a^0, z)\varphi_1 & 0 \\ 0 & 0 & 0 & \varphi_2^{-1}Y_X(a^0, z)\varphi_2 \end{bmatrix},$$

$$Y_{X'}(a^1, z) := \begin{bmatrix} 0 & Y_X(a^1, z) & 0 & 0 \\ Y_X(a^1, z) & 0 & 0 & 0 \\ 0 & 0 & \varphi_1^{-1}Y_X(a^1, z)\varphi_1 & 0 \\ 0 & 0 & 0 & \varphi_2^{-1}Y_X(a^1, z)\varphi_2 \end{bmatrix},$$

$$Y_{X'}(u^1, z)$$

$$:= \begin{bmatrix} 0 & 0 & \pi_H Y_X(\varphi_1 u^1, z)\varphi_1 & 0 \\ 0 & 0 & 0 & \pi_{H+a} Y_X(\varphi_1 u^1, z)\varphi_2 \\ \varphi_1^{-1}Y_X(\varphi_1 u^1, z) & 0 & 0 & 0 \\ 0 & \varphi_2^{-1}Y_X(\varphi_1 u^1, z) & 0 & 0 \end{bmatrix},$$

$$Y_{X'}(u^2, z)$$

$$:= \begin{bmatrix} 0 & 0 & \pi_H Y_X(\varphi_1 u^2, z)\varphi_1 & 0 \\ 0 & 0 & 0 & \pi_{H+a} Y_X(\varphi_1 u^2, z)\varphi_2 \\ \varphi_1^{-1}Y_X(\varphi_1 u^2, z) & 0 & 0 & 0 \\ 0 & \varphi_2^{-1}Y_X(\varphi_1 u^2, z) & 0 & 0 \end{bmatrix}.$$
This contradiction comes from the assumption that $H$ obtain $\psi$. Let Theorem 3.32.

Theorem 3.31. Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Then (1) For every non-zero $\alpha \in D$, the subcode $C_\alpha$ of $C$ contains a doubly even self-dual subcode w.r.t. $\alpha$. (2) $C$ is even, every codeword of $D$ has a weight divisible 8, and $D \subset C \subset D^\perp$.

Proof: (1) follows from Proposition 3.30 since $V^0 \oplus V^\alpha$ is a framed sub VOA of $V$ for any non-zero $\alpha \in D$. (2) follows from (1), since a self-dual subcode of $C_\alpha$ w.r.t. $\alpha$ always contains the codeword $\alpha$.

As a corollary, we can also prove the following theorem.

Theorem 3.32. Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Then $V = \oplus_{\alpha \in D} V^\alpha$ is a $D$-graded simple current extension of the code VOA $V^0 = V_C$.

Proof: Follows immediately from Theorem 3.31 and Corollary 3.25.
Corollary 3.33. For a positive integer $n$, the number of non-isomorphic framed VOAs with fixed central charge $n/2$ is finite.

Proof: By Theorem 3.31, every framed VOA is a simple current extension of a code VOA. A code VOA is uniquely determined by its structure code by Proposition 3.3, and it has finitely many irreducible representations as is rational. In particular, there are finitely many inequivalent simple current modules over a code VOA. Therefore, the number of non-isomorphic framed VOAs of given central charge is finite by the uniqueness of a structure of a simple current extension in Proposition 3.3.

By Theorem 3.32, we can immediately classify all irreducible (both untwisted and $\mathbb{Z}_2$-twisted) modules over a framed VOA.

Corollary 3.34. Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA with structure codes $(C, D)$. Let $W$ be an irreducible $V^0$-module. Then there exists unique $\eta \in \mathbb{Z}_2^n$ up to $D^\perp$ such that $W$ can be uniquely extended to an irreducible $\tau_\eta$-twisted $V$-module which is given by $V \boxtimes_{V^0} W$ as a $V^0$-module. In particular, every irreducible untwisted $V$-module is $D$-stable.

Proof: Let $\beta \in C^\perp$ be the $1/16$-word of $W$. Since all $V^\alpha$, $\alpha \in D$, are simple current $V^0$-submodules, the fusion product $W^\alpha := V^\alpha \boxtimes_{V^0} W$ is again irreducible. It is clear that the binary $1/16$-word of $W^\alpha$ is $\alpha + \beta$ so that all $W^\alpha$, $\alpha \in D$, are inequivalent $V^0$-modules. Therefore, there exists a unique untwisted or $\mathbb{Z}_2$-twisted $V$-module structure on $\text{Ind}^V_{V^0} W = V \boxtimes_{V^0} W = \oplus_{\alpha \in D} W^\alpha$ by Theorem 3.4. Since any element in the dual group $D^\ast \simeq \mathbb{Z}_2^n / D^\perp$ is realized as a product $\tau_\eta$ of Miyamoto involutions associated to a codeword $\eta \in \mathbb{Z}_2^n$, $\text{Ind}^V_{V^0} W$ is indeed a $\tau_\eta$-twisted $V$-module.

Remark 3.35. By the corollary above, we also see that every untwisted irreducible $V$-module is $D$-stable so that we can easily compute the fusion rules by using Proposition 3.5.

Corollary 3.36. ([DGH, M3]) A framed VOA $V$ with structure codes $(C, D)$ is holomorphic if and only if $C = D^\perp$.

Proof: That a framed VOA having a structure code $(D^\perp, D)$ is holomorphic is proved in [M3] by showing that every module contains a vacuum-like vector (cf. [Li1]). The converse is also proved in [DGH] by using modular forms. Here we give another, rather representation-theoretical proof. Let $V$ be a holomorphic framed VOA with structure codes $(C, D)$ and the $1/16$-word decomposition $V = \oplus_{\alpha \in D} V^\alpha$. Take any codeword $\delta \in D^\perp$. By the previous corollary, a $V_C$-module $M_C(0, \delta)$ can be uniquely extended to either an untwisted or $\mathbb{Z}_2$-twisted $V$-module. As a $V^0$-module, it is given by an induced module

$$V \boxtimes_{V_C} M_C(0, \delta) = \bigoplus_{\alpha \in D} V^\alpha \boxtimes_{V_C} M_C(0, \delta).$$

By Lemma 3.21, the top weight of $V^\alpha$ and that of $V^\alpha \boxtimes_{V_C} M_C(0, \delta)$ are congruent modulo $\mathbb{Z}$ for all $\alpha \in D$. Therefore, the induced module $V \boxtimes_{V_C} M_C(0, \delta)$ is an irreducible untwisted $V$-module and thus isomorphic to $V$ itself, as $V$ is holomorphic. Then by considering $1/16$-word decomposition we see that $M_C(0, \delta) = V^0 = M_C(0, 0)$. Therefore, $\delta \in C$ by Lemma 3.15 and hence $D^\perp = C$. 

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### 3.4.2 Construction of a framed VOA

In [M3, Y2], certain constructions of a framed VOA are discussed. Assume the following:

1. \((C, D)\) is a pair of even linear codes of \(\mathbb{Z}_2^n\) such that
   - (1-i) \(C \subset D^\perp\),
   - (1-ii) for each \(\alpha \in D\), there is a subcode \(E^\alpha \subset C_\alpha\) such that \(E^\alpha\) is a direct sum of the [8,4,4]-Hamming codes.

2. \(V^0\) is a code VOA associated to \(C\).

3. \(\{V^\alpha \mid \alpha \in D\}\) is a set of irreducible \(V^0\)-modules such that
   - (3-i) the 1/16-word of \(V^\alpha\) is \(\alpha\),
   - (3-ii) all \(V^\alpha\), \(\alpha \in D\), have integral top weights,
   - (3-iii) the fusion product \(V^\alpha \boxtimes_{V^0} V^\beta\) contains \(V^{\alpha+\beta}\) for all \(\alpha, \beta \in D\).

Then it is shown in [M3, Y2] that the space \(V := \oplus_{\alpha \in D} V^\alpha\) forms a framed VOA with structure codes \((C, D)\). In stead of the condition (1-ii), assume that

\[(1-ii')\] For each \(\alpha \in D\), \(C_\alpha\) contains a doubly even self-dual subcode w.r.t. \(\alpha\).

Then we have already shown in Lemma 3.27 that \(V^0 \oplus V^\alpha\) forms a framed VOA. So by the extension property of simple current extensions in Theorem 3.7, we can again show that

\[V := \oplus_{\alpha \in D} V^\alpha\]

forms a framed VOA with structure codes \((C, D)\) under the other conditions. The key idea in [M3, Y2] is to use a special symmetry of the code VOA associated to the [8,4,4]-Hamming code to form a minimal \(\mathbb{Z}_2\)-graded extension \(V^0 \oplus V^\alpha\). Thanks to Lemma 3.27, we can transcend this step without the [8,4,4]-Hamming code so that we have the following generalization.

**Theorem 3.37.** With reference to the conditions (1)-(3) above, assume the condition (1-ii') instead of (1-ii). Then \(V := \oplus_{\alpha \in D} V^\alpha\) forms a framed VOA with structure codes \((C, D)\).

**Proof:** Let \(\{\alpha^i \mid 1 \leq i \leq r\}\) be a linear basis of \(D\) and set \(D^{(i)} := \text{Span}_{\mathbb{Z}_2}\{\alpha_j \mid 1 \leq j \leq i\}\). By induction on \(i\), we show that the space \(V[i] := \oplus_{\alpha \in D^{(i)}} V^\alpha\) forms a framed VOA with structure codes \((C, D^{(i)})\). The case \(i = 0\) is trivial, and the case \(i = 1\) is done in Lemma 3.27. Now assume that \(V[i]\) forms a framed VOA for \(i \geq 1\). Then we have two simple current extensions \(V[i] = \oplus_{\alpha \in D^{(i)}} V^\alpha\) and \(V^0 \oplus V^{\alpha+1}\). Applying Theorem 3.7 to a set \(\{V^\alpha \mid \alpha \in D^{(i+1)}\}\) of simple current \(V^0\)-modules, we obtain a \(D^{(i+1)}\)-graded simple current extension \(V[i+1] = \oplus_{\alpha \in D^{(i+1)}} V^\alpha\) of \(V^0\). Repeating this, finally we shall obtain the desired framed VOA structure on \(V[r] = \oplus_{\alpha \in D} V^\alpha\).

We can also generalize Theorem 7.4.9 of [Y2] as follows:

**Theorem 3.38.** Let \(V := \oplus_{\alpha \in D} V^\alpha\) be a framed VOA with structure codes \((C, D)\). For any even subcode \(E\) such that \(C \subset E \subset D^\perp\), the space

\[\text{Ind}_E^V := \bigoplus_{\alpha \in D} \text{Ind}_{V^\alpha}^E \bigoplus_{\alpha \in D} V^\alpha \boxtimes_{V^\alpha} V^\alpha\]

forms a framed VOA with structure codes \((E, D)\).
Proof: The idea of the proof is almost the same as that of Theorem 7.4.9 of [Y2]. Let \{γ^i + C \mid 1 \leq i \leq r\} be a transversal of \(E/C\). It is clear that \(V_E = \oplus_{i=1}^r V_{C+γ^i}\) is an irreducible \(E/C\)-graded simple current extension of \(V_C\) by Proposition 3.3. Indeed, we can construct a holomorphic framed VOA starting from an irreducible untwisted \(V_C\)-module. For this, it suffices to show that \(V_{C+γ^i} \boxtimes_{V_C} V_α, 1 \leq i \leq r,\) are inequivalent \(V_C\)-modules. Assume \(V_{C+γ^i} \boxtimes_{V_C} V_α \simeq V_{C+δ} \boxtimes_{V_C} V_α\). It follows from a given framed VOA structure and Theorem 3.31 that \(V_α \boxtimes_{V_C} V_α = V^0 = V_C\). Since the fusion product is commutative and associative, multiplying \(V_α\), one has

\[
V_{C+γ^i} = V_{C+γ^i} \boxtimes_{V_C} V_α \boxtimes_{V_C} V_α = V_{C+δ} \boxtimes_{V_C} V_α \boxtimes_{V_C} V_α = V_{C+δ}
\]

in the fusion algebra. Thus, \(γ \equiv δ \mod C\) and hence all \(V_{C+γ^i} \boxtimes_{V_C} V_α, 1 \leq i \leq r,\) are inequivalent \(V_C\)-modules. Since \(E \subset D^⊥\), the top weight of \(V_α\) and that of \(V_{C+γ^i} \boxtimes_{V_C} V_α\) are congruent modulo \(Z\) by Lemma 3.21. Therefore, \(V_α\) is uniquely extended to an irreducible untwisted \(V_E\)-module \(\text{Ind}_{V_C} V_α = V_E \boxtimes_{V_C} V_α\) by Theorem 3.4. Since all \(\text{Ind}_{V_C} V_α, α \in D,\) are \(E/C\)-stable \(V_E\)-module, we have the fusion rule

\[
(\text{Ind}_{V_C} V_α) \boxtimes_{V_E} (\text{Ind}_{V_C} V_β) \simeq \text{Ind}_{V_C} \left( V_α \boxtimes_{V_C} V_β \right) \simeq \text{Ind}_{V_C} V_α + V_β
\]

by Proposition 3.3. Therefore, the space

\[
\text{Ind}_{V_C} V = \bigoplus_{α \in D} \text{Ind}_{V_C} V_α
\]

forms a framed VOA with structure codes \((E, D)\) by Theorem 3.37.

By Theorem 3.31 its corollaries and Theorem 3.37, a pair of structure codes of a holomorphic framed VOA is given by \((C, C^⊥)\) satisfying the following conditions.

(1) The length of \(C\) is divisible by 16.

(2) \(C\) is even, every codeword of \(C^⊥\) has a weight divisible by 8, and \(C^⊥ \subset C\).

(3) For any \(α \in C^⊥\), the subcode \(C_α\) of \(C\) contains a doubly even self-dual subcode w.r.t. \(α\).

For simplicity, a code \(C\) is called \(F\)-admissible if it satisfies conditions (1)-(3) above. Indeed, we can construct a holomorphic framed VOA starting from an \(F\)-admissible code.

Let \(C\) be an \(F\)-admissible code. Then all one vector \(1\) is contained in \(C\). Since \(n = \text{wt}(1)\) is divisible by 16, the irreducible \(V_C\)-module \(M_C(1, 0)\) has the integral top weight. Denote \(V^1 := M_C(1, 0)\). Note that \(V^1\) is a self-dual simple current \(V_C\)-module.

Lemma 3.39. Let \(C\) be an \(F\)-admissible code. For \(α \in C^⊥, α \not\in \{0, 1\}\), there exists an irreducible \(V_C\)-module \(M_C(α, δ_α)\) such that both \(M_C(α, δ_α)\) and the fusion product \(V^1 \boxtimes_{V_C} M_C(α, δ_α)\) have integral top weights.

Proof: Clearly, we can find a codeword \(δ \in \mathbb{Z}_2^n\) such that \(M_C(α, δ)\) has an integral top weight. The 1/16-word of the fusion product \(V^1 \boxtimes_{V_C} M_C(α, δ)\) is then \(1 + α\) and thus \(V^1 \boxtimes_{V_C} M_C(α, δ)\) is isomorphic to \(M_C(1 + α, δ')\) for some \(δ' \in \mathbb{Z}_2^n\). The top weight of
$M_C(1 + \alpha, \delta')$ is in either $\mathbb{Z}$ or $\mathbb{Z} + 1/2$. If the weight is in $\mathbb{Z} + 1/2$, we can take a codeword $\delta'' \in (\mathbb{Z}_{2^2})_\alpha$ such that $\delta''$ is odd. In this case, $M_C(1 + \alpha, \delta' + \delta'')$ has an integral top weight. Since $\text{supp}(\delta'') \subseteq \text{supp}(\alpha)$, the top weight of $M_C(\alpha, \delta + \delta'')$ is the same as that of $M_C(\alpha, \delta)$, which is also integral. Moreover, by Lemma 3.21

$$V^1 \boxtimes_{V_C} M_C(\alpha, \delta + \delta'') = V^1 \boxtimes_{V_C} M_C(\alpha, \delta) \boxtimes_{V_C} M_C(0, \delta'')$$

$$= M_C(1 + \alpha, \delta') \boxtimes_{V_C} M_C(0, \delta'')$$

$$= M_C(1 + \alpha, \delta' + \delta'')$$

Now set $\delta_\alpha = \delta + \delta''$ and we have the desired result. 

**Proposition 3.40.** Let $C$ be an $F$-admissible code and $D$ a proper subcode of $C$ containing $1$. Suppose that we have a framed VOA $V = \bigoplus_{\alpha \in D} V^\alpha$ with structure codes $(C, D)$ and $V^1 = M_C(1, 0)$. Then for $\beta \in C^\perp \setminus D$, there exist a self-dual simple current $V$-module $W$ such that the $1/16$-word decomposition of $W$ is $W = \bigoplus_{\alpha \in D} W^{\alpha+\beta}$ and $\tilde{V} = V \oplus W$ is a framed VOA with structure codes $(C, D + \langle \beta \rangle)$. 

**Proof:** Set $W^\beta := M_C(\beta, \delta_\beta)$ such that both $W$ and $V^1 \boxtimes_{V_C} W$ are of integral weight (cf. Lemma 3.39). Since $\beta \in C^\perp$, it follows from Condition (3) that $W^\beta$ is a self-dual simple current $V_C$-module. Then the induced module $\text{Ind}^V_C W^\beta = \bigoplus_{\alpha \in D} V^\alpha \boxtimes_{V_C} W^\beta$ is an irreducible $\tau_\eta$-twisted $V$-module by Corollary 3.34. If $\eta \in D^\perp$, then the space $V \oplus \text{Ind}^V_C W^\beta$ forms a framed VOA with structure codes $(C, D + \langle \beta \rangle)$ by Theorem 3.37.

If $\eta \notin D^\perp$, $\tau_\eta \neq 1$. Set $D^+ := \{\alpha \in D \mid \langle \alpha, \eta \rangle = 0\}$ and $D^- := \{\alpha \in D \mid \langle \alpha, \eta \rangle = 1\}$. Then $D = D^+ \cup D^-$ and $D^\perp \neq \emptyset$. By our choice of $W^\beta$, the all one vector $1$ is in $D^+$ so that $\eta$ is an even codeword. We set

$$V^\pm := \bigoplus_{\alpha \in D^\pm} V^\alpha, \quad W^\pm := \bigoplus_{\alpha \in D^\pm} V^\alpha \boxtimes_{V_C} W^\beta.$$

Then all $V^\pm, W^\pm$ are irreducible $V^+$-modules. The top weight of $W^+$ is integral but the top weight of $W^-$ is in $\mathbb{Z} + 1/2$. We will deform $W^-$ so that it has an integral top weight, also.

Since $[(D^+)\perp \cap \langle \beta \rangle, D^\perp \cap \langle \beta \rangle] = 2$, there exists a codeword $\gamma \in (D^+)\perp \cap \langle \beta \rangle$ such that $\langle \gamma, D^- \rangle = 1$. Then it follows from Corollary 3.34 that

$$\tilde{W}^\pm := V_{C+\gamma} \boxtimes_{V_C} W^\pm$$

are irreducible untwisted $V^+$-modules. Moreover, by our choice of $\gamma$, both of $\tilde{W}^\pm$ have integral top weights since the top weight of $W^+$ is congruent to $\langle \gamma, \gamma + \beta \rangle/2$ modulo $\mathbb{Z}$, whereas that of $\tilde{W}^-$ is to $\langle \gamma, \beta + \gamma + D^- \rangle/2 + 1/2$ modulo $\mathbb{Z}$. Therefore, by Theorem 3.37, we have a framed VOA structure on

$$\tilde{V} := V^+ \oplus V^- \oplus \tilde{W}^+ \oplus \tilde{W}^-$$

with structure codes $(C, D + \langle \beta \rangle)$. Now setting $W := \tilde{W}^+ \oplus \tilde{W}^-$, we have the desired extension of $V$. This completes the proof. 

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Remark 3.41. In the proof above, we can construct another extension of $V^+$ which also has the structure codes $(C, D + \langle \beta \rangle)$ in the following way. Take a codeword $\gamma \in D^\perp$ with $\langle \gamma, \beta \rangle = 1$, which is possible as $[D^\perp : D^\perp \cap \langle \beta \rangle^\perp] = 2$, and set $\tilde{V}^- = V_{C+\gamma} \boxtimes V_C$, $V^-$ and $W^- = V_{C+\gamma} \boxtimes V_C W^-$. Then one can similarly verify that the space $V^+ \oplus V^- \oplus W^+ \oplus \tilde{W}^-$ also forms a framed VOA with structure codes $(C, D + \langle \beta \rangle)$.

Theorem 3.42. The pair $(C, C^\perp)$ is a structure codes of a holomorphic framed VOA if and only if $C$ is $F$-admissible.

Proof: Let $\{\alpha_1, \ldots, \alpha_r\}$ be a linear basis of $C^\perp$ with $\alpha_1 = 1$ and set $D[i] := \text{Span}_{z_2} \{\alpha_j \mid 1 \leq j \leq i\}$ for $1 \leq i \leq r$. By Lemma 3.27, we can form a framed VOA $V[1] := V_C \oplus M_C(1, 0)$ with structure codes $(C, D[1])$. By Proposition 3.40, we can construct a framed VOA $V[2]$ with structure codes $(C, D[2])$ which is a $\mathbb{Z}_2$-graded simple current extension of $V[1]$. Recursively, we can construct a $\mathbb{Z}_2$-graded simple current extension $V[i + 1]$ of $V[i]$ which have a structure codes $(C, D[i + 1])$ and we will obtain a holomorphic framed VOA $V[r]$ with structure codes $(C, D[r]) = (C, C^\perp)$.

Remark 3.43. By Theorem 3.42, the classification of holomorphic framed VOAs is almost reduced to the classification of $F$-admissible codes $C$. The $F$-admissible conditions, especially (2) and (3), give quite strong restrictions on a code $C$. Roughly speaking, $C$ must be much bigger than its dual $C^\perp$ by the condition (3). In addition, if we assume that the minimum weight of $C$ is greater than 2, then the corresponding framed VOA may have a finite full automorphism group (cf. [LSY, Corollary 3.9]). It seems possible to classify all such codes $C$ if the length is small. The most interesting (and the first non-trivial) case would be the classification of $c = 24$ holomorphic framed VOAs. In fact, one can prove that the moonshine vertex operator algebra $V^\natural$ is the unique holomorphic framed VOA of rank 24 whose weight one subspace is trivial, which is a variant of the famous uniqueness conjecture of the moonshine vertex operator algebra proposed in [FLM] (see also [DGL]). The key point is that the structure codes of $V^\natural$ (or any holomorphic framed VOA $V$ of rank 24 and $V_1 = 0$) are closely related to those of the Leech lattice VOA $V_\Lambda$. In fact, if $V = \bigoplus_{\alpha \in C^\perp} V^\alpha$, $V^0 = V_C$, is a holomorphic framed VOA $V$ of rank 24 and $V_1 = 0$, then the minimal weight of $C$ is 4. In this case, for any $\delta \in \mathbb{Z}_2^{48}$ of weight 2, the $\tau_\delta$-orbifold VOA

$$V(\tau_\delta) = \bigoplus_{\alpha \in D} \left( V^\alpha \oplus V_{C+\delta} \boxtimes V^\alpha \right), \quad D = \{\alpha \in C^\perp \mid \langle \alpha, \delta \rangle = 0\},$$

is isomorphic to the Leech lattice $V_\Lambda$ and $(C\cup(\delta+C), D)$ will be the structure codes of $V_\Lambda$. Note that the weight one subspace of $V(\tau_\delta)$ forms an abelian Lie algebra with respect to the bracket $[a, b] = a_\delta b$. We will give more details on this point and the possible structure codes of $V^\natural$ in our next work.

3.5 Frame stabilizers and order four symmetries

In Section 5, we have seen that structure codes $(C, D)$ of a framed VOA $V = \bigoplus_{\alpha \in D} V^\alpha$ satisfy certain duality conditions. The main property is that for any $\alpha \in D$, the subcode $C_\alpha$ contains a doubly even self-dual subcode w.r.t. $\alpha$ and $V^\alpha$ is a simple current $V_0^\alpha$-module. However, it is shown in Corollary 3.25 that $V^\alpha$ is a simple current module.
without the assumption on doubly even. In this section we shall discuss about the role of the doubly even property. It will turns out that by relaxing the doubly even property, we can obtain a refinement of the 1/16-word decomposition and define an automorphism of order four in the pointwise frame stabilizer.

We begin by defining the frame stabilizer and the pointwise frame stabilizer of a framed VOA.

**Definition 3.44.** Let $V$ be a framed VOA with an Ising frame $F = \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n)$. The frame stabilizer of $F$ is the subgroup of all automorphisms of $V$ which stabilizes the frame $F$ setwise. The pointwise frame stabilizer is the subgroup of $\text{Aut}(V)$ which fixes $F$ pointwise. The frame stabilizer and the pointwise frame stabilizer of $F$ are denoted by $\text{Stab}_V(F)$ and $\text{Stab}_{pt}^V(F)$, respectively.

Let $(C,D)$ be the structure code of $V$ with respect to $F$, i.e.,

$$V = \bigoplus_{\alpha \in D} V^\alpha, \quad \tau(V^\alpha) = \alpha \quad \text{and} \quad V^0 = V_C.$$

For any $\theta \in \text{Stab}_{pt}^V(F)$, it is easy to see that $\tau_{\theta e^i} = \tau_{\theta e^n} = \theta \tau_{e^i} \theta^{-1}$ and thus $\theta$ centralizes $\langle \tau_{e^i}, \ldots, \tau_{e^n} \rangle$. Therefore, the group $\tau(\mathbb{Z}_2^n) = \langle \tau_{e^1}, \ldots, \tau_{e^n} \rangle$ generated by the $\tau$-involutions is a central subgroup of $\text{Stab}_{pt}^V(F)$ isomorphic to $\mathbb{Z}_2^n$. In addition, we have $\theta V^\alpha = V^\alpha$ for all $\alpha \in D$ and hence $\theta|_{V^0}$ is an automorphism of $V^0$.

The following results can be proved easily using the fusion rules.

**Lemma 3.45.** Let $V = \oplus_{\alpha \in D} V^\alpha$ be a framed VOA.

1. Let $\phi \in \text{Aut}(V^0)$ such that $\phi|_{V^0} = \text{id}_{F}$. Then $\phi \in \sigma(\mathbb{Z}_2^n) = \langle \sigma_{e^1}, \ldots, \sigma_{e^n} \rangle$.

2. Let $g \in \text{Aut}(V)$ such that $g|_{V^0} = \text{id}_{V^0}$. Then $g \in \tau(\mathbb{Z}_2^n) = \langle \tau_{e^1}, \ldots, \tau_{e^n} \rangle$.

**Proof:** (1) Consider the 1/2-word decomposition

$$V^0 = \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in C} L(\frac{1}{2}, \beta_1/2) \otimes \cdots \otimes L(\frac{1}{2}, \beta_n/2)$$

of $V^0$ as an $F = \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n)$-module. Since $\phi|_{V^0} = \text{id}_{F}$, it follows from Schur’s lemma that $\phi|_{V^0}$ acts on $L(\frac{1}{2}, \beta_1/2) \otimes \cdots \otimes L(\frac{1}{2}, \beta_n/2)$ by a non-zero scalar $a_{\beta}$ for each $\alpha \in C$. Moreover, it follows from the fusion rules of $L(\frac{1}{2}, 0)$-modules in (3.6) that $a_{\alpha}a_{\beta} = a_{\alpha+\beta}$ for all $\alpha, \beta \in D$. Thus the association $C \ni \alpha \mapsto a_{\alpha} \in \mathbb{C}$ defines a character of $C$ and hence there is a codeword $\xi \in \mathbb{Z}_2^n$ such that $a_{\alpha} = (-1)^{\langle \xi, \alpha \rangle}$. Now it is easy to see that $\phi|_{V^0}$ is realizable as a product of $\sigma_{e^i}$, $1 \leq i \leq n$, that is, $\phi|_{V^0} = \sigma_{\xi}$.

(2) Since each $V^\alpha$, $\alpha \in D$, is an irreducible $V^0$-module, it follows from Schur’s lemma that $g$ acts on $V^\alpha$ by a non-zero scalar $t_{\alpha} \in \mathbb{C}$. Then again by the fusion rules of $L(\frac{1}{2}, 0)$-modules in (3.6) we have $t_{\alpha}t_{\beta} = t_{\alpha+\beta}$ so that the map $\alpha \mapsto t_{\alpha}$ defines a character of $D$. Therefore, there exists a codeword $\eta \in \mathbb{Z}_2^n$ such that $g = \tau_\eta \in \langle \tau_{e^1}, \ldots, \tau_{e^n} \rangle$.

As a corollary, we have the following theorem.

**Theorem 3.46.** Let $V$ be a framed VOA with an Ising frame $F = \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n)$. For any $\theta \in \text{Stab}_{pt}^V(F)$, there exist $\xi$ and $\eta \in \mathbb{Z}_2^n$ such that

$$\theta|_{V^0} = \sigma_{\xi} \quad \text{and} \quad \theta^2 = \tau_\eta.$$

In particular, we have $\theta^4 = 1$. 73
Let $\theta \in \text{Stab}^\eta_V(F)$. Then $\theta|_{V^0} = \sigma_\xi$ for some $\xi \in \mathbb{Z}_2^n$. That means $\theta$ is a lifting of a $\sigma$-involution on $V^0$ to whole framed VOA $V$. In this section, we shall give a necessary and sufficient condition on whether a $\sigma$-involution $\sigma_\xi$ can be extended to the whole $V$. Our method is based on the representation theory of code VOAs developed in Section 4 and 5.

First let us consider $\theta \in \text{Stab}^\eta_V(F)$ such that $\theta|_{V^0} = \sigma_\xi \neq \text{id}_{V^0}$, i.e., $\xi \notin C^\perp$. Set $C^0 := \{\alpha \in C \mid \langle \xi, \alpha \rangle = 0\}$ and $C^1 := \{\alpha \in C \mid \langle \xi, \alpha \rangle = 1\}$. Then $C^0$ is a subcode of $C$, $[C : C^0] = 2$ and $C = C^0 \sqcup C^1$. Note also that $V_{C^0}$ is fixed by $\theta$ and $\theta$ acts by $-1$ on $V_{C^1}$. In other words, $V^0 = V_{C^0} \oplus V_{C^1}$ is the eigenspace decomposition of $\theta$ on $V^0$.

Now assume that $\theta^2 = \tau_\eta$ for some $\eta \in \mathbb{Z}_2^n$. For each non-zero $\alpha \in D$, it is clear that $V^0 \oplus V^\alpha$ is a subalgebra of $V$ and $\theta$ stabilizes $V^0 \oplus V^\alpha$. If $\alpha \in D \cap \langle \eta \rangle^\perp$, then $\theta^2$ acts as an identity on $V^0 \oplus V^\alpha$ and the eigenvalues of $\theta$ on $V^\alpha$ are $\pm 1$. Let $V^\alpha_+$ and $V^\alpha_-$ be the eigenspaces of $\theta$ with eigenvalues $+1$ and $-1$, respectively. Note that both $V^\alpha_+$ are non-zero inequivalent irreducible $V^0+$-submodules. If $V^\alpha_+ = 0$, then $V^\alpha_- \cdot V^\alpha_- = V^\alpha_+ = 0$, which contradicts Proposition 11.9 of [DL]. Since the subalgebra $V^0 \oplus V^\alpha = V^{0+} \oplus V^{-} \oplus V^\alpha_+ \oplus V^\alpha_-$ affords a faithful action of a group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of order 4, $V^\alpha$ are inequivalent irreducible $V^0+$-submodules by the quantum Galois theory.

If $\alpha \in D \setminus \langle \eta \rangle^\perp$, then $\theta^2 = -1$ on $V^\alpha$ and the eigenvalues of $\theta$ on $V^\alpha$ are $\pm \sqrt{-1}$. Let $V^\alpha_\pm$ be the eigenspace of $\theta$ of eigenvalues of $\pm \sqrt{-1}$ on $V^\alpha$. Then $V^\alpha = V^\alpha_+ \oplus V^\alpha_-$. $V^\alpha$ are again non-zero inequivalent irreducible $V^0+$-submodules. The argument above actually shows that $\theta$ is of order 2 if and only if $\langle \eta, D \rangle = 0$, i.e., $\eta \in D^\perp$.

By the observation above, we have

**Lemma 3.47.** For any $\alpha \in D$, let $V^\alpha_\pm$ be defined as above. Then the dual $V^{0+}$-module $(V^{\alpha \pm})^*$ is isomorphic to $V^{\alpha \pm}$ if and only if $\alpha \notin \langle \eta \rangle^\perp$. Otherwise, $(V^{\alpha \pm})^*$ is isomorphic to $V^{\alpha \mp}$.

**Proof:** Since any framed VOA is self-dual, the sub VOA $V^0 \oplus V^\alpha$ of $V$ is also self-dual. Since $V^\alpha_+ \cdot V^\alpha_- = V^{0+}$ if and only if $\alpha \notin \langle \eta \rangle^\perp$, the duality is as in the assertion.

We have shown that for any $\theta \in \text{Stab}^\eta_V(F) \setminus \tau(\mathbb{Z}_2^n)$, $|\theta| = 2$ if and only if all irreducible $V^0+$-submodules of $V$ are self-dual, and otherwise $|\theta| = 4$. We rewrite this condition in terms of the structure codes as follows.

**Lemma 3.48.** Let $\theta \in \text{Stab}^\eta_V(F)$ such that $\theta|_{V^0} = \sigma_\xi$ and $\theta^2 = \tau_\eta$ for some $\xi \in \mathbb{Z}_2^n \setminus C^\perp$ and $\eta \in \mathbb{Z}_2^n$.

(1) For $\alpha \in D \cap \langle \eta \rangle^\perp$, $(C^0)_\alpha$ contains a doubly even self-dual subcode w.r.t. $\alpha$.

(2) For $\alpha \in D \setminus \langle \eta \rangle^\perp$, $(C^0)_\alpha$ contains a self-dual subcode w.r.t. $\alpha$, but $(C^0)_\alpha$ does not contain any doubly even self-dual subcode w.r.t. $\alpha$.

**Proof:** (1) For $\alpha \in D \cap \langle \eta \rangle^\perp$, let $V^\alpha = V^{\alpha +} \oplus V^{\alpha -}$ be the eigenspace decomposition such that $\theta$ acts on $V^{\alpha \pm}$ by $\pm 1$. In this case the subspace $V^{0+} \oplus V^{\alpha +}$ forms a framed sub VOA of $V$. By Proposition 3.30, $(C^0)_\alpha$ contains a doubly even self-dual subcode w.r.t. $\alpha$.

(2) For $\alpha \in D \setminus \langle \eta \rangle^\perp$, let $V^\alpha = V^{\alpha +} \oplus V^{\alpha -}$ be the eigenspace decomposition such that $\theta$ acts on $V^{\alpha \pm}$ by $\pm \sqrt{-1}$. In this case the restriction of $\theta$ on the sub VOA $V^{0+} \oplus V^{\alpha +}$ forms a framed sub VOA of $V$.

$$V^0 \oplus V^\alpha = V^{0+} \oplus V^{\alpha -} \oplus V^{\alpha +} \oplus V^{\alpha -}$$

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is of order 4. By the quantum Galois theory [?], \(V^{α^+}\) and \(V^{α^-}\) are inequivalent irreducible \(V^{0+} = V_{co}\)-modules. By Lemma 3.47, \(V^{α^+}\) and \(V^{α^-}\) are dual to each other. Therefore, by Proposition 3.18, any maximal self-orthogonal subcode of \((C^0)_α\) is not doubly even. Let \(H\) be a doubly even self-dual subcode of \(C_α\) w.r.t. \(α\) and \(H^0\) a maximal self-orthogonal subcode of \((C^0)_α\). Since \((C^0)_α\) does not contain a doubly even self-dual subcode w.r.t. \(α\), \((C^0)_α\) is a proper subgroup of \(C_α\) so that \([C_α : (C^0)_α]\) = 2. It follows from Theorem 3.12 that \(V^α\) consists of \([C : C_α]\) inequivalent \(V(0)\)-submodules with each multiplicity \([C_α : H]\). Similarly, each of \(V^{α±}\) consists of \([C^0 : (C^0)_α]\) inequivalent irreducible \(F\)-submodules with each multiplicity \([C^0_α : H^0]\). Since \(V^{α^+}\) and \(V^{α^-}\) are dual to each other, they are isomorphic as \(F\)-modules. Therefore, by counting multiplicity of irreducible \(F\)-submodules of \(V^α\) and \(V^{α±}\), one has \([C_α : H]\) = 2\([C^0_α : H^0]\). Combining with \([C_α] = 2\|[C^0_α]\|\), we obtain \(|H| = |H^0| = 2^{\text{wt}(α)/2}\). Therefore, \(H^0\) is a self-dual subcode of \((C^0)_α\) w.r.t. \(α\).

**Lemma 3.49.** Let \(C\) be an even code and \(β ∈ C^⊥\). Assume that \(V = V_C \oplus M_C(β, γ)\) forms a framed VOA and \(C\) contains a subcode \(E\) with index two such that \(E\) contains a self-dual subcode w.r.t. \(β\). Then \(V\) decomposes into a direct sum of four inequivalent simple current \(V_E\)-submodules

\[V = M_E(0, 0) \oplus M_E(0, δ) \oplus M_E(β, γ) \oplus M_E(β, γ + δ),\]

where \(δ ∈ C\) is such that \(C = E \sqcup (E + δ)\). Moreover, all irreducible \(V_E\)-submodules of \(V\) are self-dual if and only if \(E\) contains a doubly even self-dual subcode w.r.t. \(α\).

**Proof:** Let \(δ ∈ C\) be such that \(C = E \sqcup (E + δ)\). Then the decomposition \(V_C = M_E(0, 0) \oplus M_E(0, δ)\) is obvious. The decomposition \(M_C(β, γ) = M_E(β, γ) \oplus M_E(β, γ + δ)\) also follows from Corollary 3.16. That all irreducible \(V_E\)-submodules are simple currents follows from Corollary 3.23. If \(E\) contains a doubly even self-dual subcode w.r.t. \(β\), then all irreducible \(V_E\)-submodules of \(V\) are self-dual by Proposition 3.18. Conversely, if all irreducible \(V_E\)-submodules of \(V\) are self-dual, then \(V_E \oplus M_E(β, γ)\) forms a sub VOA of \(V\) so that \(E\) contains a doubly even self-dual subcode w.r.t. \(β\) by Proposition 3.30. This completes the proof.

**Notation.** For \(α = (α_1, \ldots, α_n), β = (β_1, \ldots, β_n) ∈ \mathbb{Z}_2^n\), we define

\[α · β := (α_1 β_1, \ldots, α_n β_n) ∈ \mathbb{Z}_2^n.\]

That is, the product \(α · β\) is taken in the ring \(\mathbb{Z}_2^n\). Note that \(α · β ∈ (\mathbb{Z}_2^n)_α \cap (\mathbb{Z}_2^n)_β\).

**Theorem 3.50.** Let \(V\) be a framed VOA with an Ising frame \(ω = e^1 + \cdots + e^n\) and let \((C, D)\) be the structure code with respect to \(F = Vir(e^1) ⋉ \cdots ⋉ Vir(e^n)\). For a codeword \(ξ ∈ \mathbb{Z}_2^n \setminus C^⊥\), there exists \(θ ∈ \text{Stab}_V^pt(F)\) such that \(θ|_{V_0} = σ_ξ\) if and only if \(α · ξ ∈ C\) for all \(α ∈ D\). Moreover, \(|θ| = 2\) if and only if \(\text{wt}(α · ξ) ≡ 0 \mod 4\) for all \(α ∈ D\), and otherwise \(|θ| = 4\).

**Proof:** Suppose \(θ ∈ \text{Stab}_V^pt(F)\) is such that \(θ|_{V_0} = σ_ξ\) with \(ξ ∈ \mathbb{Z}_2^n \setminus C^⊥\). Then by Lemma 3.48, \((C^0)_α\) contains a self-dual subcode w.r.t. \(α\) for any \(α ∈ D\). Since

\[(C^0)_α = \{β ∈ C \mid ⟨β, ξ⟩ = 0 \text{ and } \text{supp}(β) ⊂ \text{supp}(α)\}\]
and \( \langle \beta, \xi \rangle = \langle \beta, \alpha \cdot \xi \rangle = 0 \) for all \( \beta \in (C^0)_\alpha \), \( \alpha \cdot \xi \in ((C^0)_\alpha)^\perp \) for all \( \alpha \in D \). Therefore, \( \alpha \cdot \xi \) is contained in all self-dual subcodes of \( C^0 \) w.r.t. \( \alpha \) and hence \( \alpha \cdot \xi \in (C^0)_\alpha \subset C \) as claimed.

Conversely, assume that a codeword \( \xi \in \mathbb{Z}^n_2 \setminus C^\perp \) satisfies \( \alpha \cdot \xi \subset C \) for all \( \alpha \in D \). Then \( C^0 = C \cap \langle \xi \rangle^\perp \) is a proper subcode of \( C \) with index 2. By definition, \( \alpha \cdot \xi \in ((C^0)_\alpha)^\perp \) for all \( \alpha \in D \). Therefore, any maximal self-orthogonal subcode of \( (C^0)_\alpha \) contains \( \alpha \cdot \xi \). Set \( D^0 := \{ \alpha \in D \mid \text{wt}(\alpha \cdot \xi) \equiv 0 \mod 4 \} \) and \( D^1 := \{ \alpha \in D \mid \text{wt}(\alpha \cdot \xi) \equiv 2 \mod 4 \} \). It is clear that \( D = D^0 \sqcup D^1 \). If \( \alpha \in D^0 \), then there exists a doubly even self-dual subcode of \( (C^0)_\alpha \) w.r.t. \( \alpha \). For, let \( H \) be a doubly even self-dual subcode of \( C_\alpha \), which exists by Proposition 3.30. If \( H \) is contained in \( (C^0)_\alpha \), then we are done. If not, then \( \alpha \cdot \xi \not\subset H \) and \( H \cap (C^0)_\alpha = H \cap \langle \xi \rangle^\perp \) is a subcode of \( H \) with index 2 so that

\[
(H \cap \langle \xi \rangle^\perp) \sqcup (H \cap \langle \xi \rangle^\perp + \alpha \cdot \xi)
\]
gives a doubly even self-dual subcode of \( (C^0)_\alpha \) w.r.t. \( \alpha \). Similarly, we can show that \( (C^0)_\alpha \) contains a self-dual subcode w.r.t. \( \alpha \) for any \( \alpha \in D^1 \). But in this case any self-dual subcode of \( (C^0)_\alpha \) w.r.t. \( \alpha \) is not doubly even, as it always contains \( \alpha \cdot \xi \). We have shown that for each \( \alpha \in D \), \( (C^0)_\alpha \) contains a self-dual subcode w.r.t. \( \alpha \) so that one has a \( V_{C^0} \)-module decomposition \( V^\alpha = V^{\alpha,1} \oplus V^{\alpha,2} \) such that \( V^{\alpha,p}, p = 1, 2 \), are simple current \( V_{C^0} \)-submodules by Lemma 3.49.

Let \( \{ \alpha^1, \ldots, \alpha^r \} \) be a linear basis of \( D^0 \). For each \( i, 1 \leq i \leq r \), choose an irreducible \( V_{C^0} \)-submodule \( U^{\alpha^i} \) of \( V^{\alpha^i} \) arbitrary. Then for \( \alpha = \alpha^{i_1} + \cdots + \alpha^{i_k} \in D^0 \), set

\[
U^\alpha := U^{\alpha^{i_1}} \bigotimes_{V_{C^0}} \cdots \bigotimes_{V_{C^0}} U^{\alpha^{i_k}}.
\]

Since all \( U^{\alpha^i}, 1 \leq i \leq r \), are simple current self-dual \( V_{C^0} \)-modules, \( U^\alpha \) is uniquely defined by (3.18) for all \( \alpha \in D^0 \). Note that \( U^0 = V_{C^0} \). Since \( \oplus_{\alpha \in D^0} V^\alpha \) is a sub VOA of \( V \), \( U^\alpha \) are irreducible \( V_{C^0} \)-submodules of \( V^\alpha \) for all \( \alpha \in D^0 \). Therefore, we obtain a framed sub VOA \( U := \oplus_{\alpha \in D^0} U^\alpha \) of \( V \) with structure codes \( (C^0, D^0) \). It is easy to see that \( V^\alpha = U^\alpha \oplus (V_{C^1} \bigotimes_{V_{C^0}} U^\alpha) \) for \( \alpha \in D^0 \) by Lemma 3.49.

If \( D = D^0 \), then we have \( V = U \oplus (V_{C^1} \bigotimes_{V_{C^0}} U) \) as a \( V_{C^0} \)-module. In this case we define a linear automorphism \( \theta_\xi \) on \( V \) by

\[
\theta_\xi := \begin{cases} 
1 & \text{on } U, \\
-1 & \text{on } V_{C^1} \bigotimes_{V_{C^0}} U.
\end{cases}
\]

Then it follows from Lemma 3.49 and Proposition 3.22 that \( \theta_\xi \in \text{Stab}^R_V(F) \) and \( \theta_\xi|_{V^0} = \sigma_\xi \). Therefore, \( \sigma_\xi \) can be extended to an involution on \( V \).

If \( D \neq D^0 \), then \( D = D^0 \sqcup D^1 \) with \( D^1 \neq \emptyset \). In this case, take one \( \beta \in D^1 \) and an irreducible \( V_{C^0} \)-submodule \( W^\beta \) of \( V^\beta \). Since \( W^\beta \) and all \( U^{\alpha}, \alpha \in C^0 \), are simple current \( V_{C^0} \)-modules, we have a \( V_{C^0} \)-module decomposition

\[
V^{\alpha+\beta} = (U^{\alpha} \bigotimes_{V_{C^0}} W^\beta) \oplus (V_{C^1} \bigotimes_{V_{C^0}} U^{\alpha} \bigotimes_{V_{C^0}} W^\beta)
\]
of $V^{\alpha + \beta}$ for all $\alpha \in C^0$ by Lemma 3.49. Since $(C^0)_{\alpha + \beta}$ contains no doubly even self-dual subcode w.r.t. $\alpha + \beta$, the decomposition

$$V^0 \oplus V^{\alpha + \beta} = V_{C^0} \oplus V_{C^1} \oplus (U^\alpha \boxtimes W^\beta) \oplus (V_{C^1} \boxtimes U^\alpha \boxtimes W^\beta)$$

induces an order four automorphism on a sub VOA $V^0 \oplus V^{\alpha + \beta}$ of $V$ by Lemma 3.49 and Proposition 3.22. Set

$$W := \bigoplus_{\alpha \in C^0} U^\alpha \boxtimes W^\beta.$$ 

Then we have obtained the following decomposition of $V$ as a $V_{C^0}$-module:

$$V = U \oplus (V_{C^1} \boxtimes U) \oplus W \oplus (V_{C^1} \boxtimes W).$$

We define a linear automorphism $\theta_\xi$ on $V$ by

$$\theta_\xi := \begin{cases} 
1 & \text{on } U, \\
-1 & \text{on } V_{C^1} \boxtimes V_{C^0} U, \\
\sqrt{-1} & \text{on } W, \\
-\sqrt{-1} & \text{on } V_{C^1} \boxtimes V_{C^0} W.
\end{cases}$$

Then it follows from the argument above that $\theta_\xi \in \text{Stab}_V^F$ and $\theta_\xi|_{V^0} = \sigma_\xi$. Therefore, $\sigma_\xi$ gives rise to a pointwise frame stabilizer of order 4.

Summarizing, we have shown that there exists $\theta \in \text{Stab}_V^F$ such that $\theta|_{V^0} = \sigma_\xi$. By the theorem above, we know that this is not correct; we have to take a codeword $\beta$ to satisfy $\alpha \cdot \beta \in C$ for all $\alpha \in D$.

Motivated by Theorem 3.50, we define $P := \{\xi \in \mathbb{Z}_2^n \mid \alpha \cdot \xi \in C \text{ for all } \alpha \in D\}$. It is clear that $P$ is a linear subcode of $C$. Moreover, we have

Lemma 3.52. $C^\perp \subset P$.

Proof: Let $\delta \in C^\perp$. For $\alpha \in D$, by definition one has $\langle \delta, C_\alpha \rangle = 0$ so that $\langle \alpha \cdot \delta, C_\alpha \rangle = 0$. Since $C_\alpha$ contains a self-dual subcode w.r.t. $\alpha$ by Theorem 3.31, $\alpha \cdot \delta \in C_\alpha \subset C$.

For each codeword $\xi \in P$, there exists $\theta_\xi \in \text{Stab}_V^F$ such that $\theta_\xi|_{V^0} = \sigma_\xi$ by Theorem 3.50. However, the construction of $\theta_\xi$ in the proof of Theorem 3.50 is not unique since we have to choose a linear basis of $D^0$ and irreducible $V_{C^0}$-submodules. In fact, the following holds.

Lemma 3.53. Let $\theta, \phi \in \text{Stab}_V^F$ such that $\theta|_{V^0} = \phi|_{V^0} = \sigma_\xi$. Then $\phi = \theta \tau_\eta$ for some $\eta \in \mathbb{Z}_2^n$. 77
Proof: Since \( \theta|_{V^0} = \phi|_{V^0} \), we have \( \theta^{-1}\phi|_{V^0} = \text{id}_{V^0} \). By Lemma 3.45 there exists \( \eta \in \mathbb{Z}_2^n \) such that \( \theta^{-1}\phi = \tau_\eta \) and hence \( \phi = \theta\tau_\eta \) as desired.

In other words, \( \theta_\xi \) is only determined up to a \( \tau \)-involution. We have also seen in Lemma 3.45 that \( \theta_\xi \in \tau(\mathbb{Z}_2^n) \) if and only if \( \xi \in C^{\perp} \). Since \( C^{\perp} \subseteq P \) by Lemma 3.52, the association \( \xi + C^{\perp} \mapsto \theta_\xi\tau(\mathbb{Z}_2^n) \) defines a group isomorphism between \( P/C^{\perp} \) and \( \text{Stab}^\text{pt}_\tau(F)/\tau(\mathbb{Z}_2^n) \). Therefore, we have the following central extension:

\[
1 \longrightarrow \tau(\mathbb{Z}_2^n) \cong \mathbb{Z}_2^n/D \longrightarrow \text{Stab}^\text{pt}_\tau(F) \longrightarrow \text{Stab}^\text{pt}_\tau(F)/\tau(\mathbb{Z}_2^n) \cong P/C^{\perp} \longrightarrow 1.
\]

The commutator relation in \( \text{Stab}^\text{pt}_\tau(F) \) can also be described as follows.

**Theorem 3.54.** For \( \xi^1, \xi^2 \in P \), let \( \theta_{\xi^i}, i = 1,2 \), be an extension of \( \sigma_\xi \) to \( \text{Stab}^\text{pt}_\tau(F) \). Then \( [\theta_{\xi^1}, \theta_{\xi^2}] = 1 \) if and only if \( \langle \alpha \cdot \xi^1, \alpha \cdot \xi^2 \rangle = 0 \) for all \( \alpha \in D \).

**Proof:** Since the case \( \theta_{\xi^1} \in \theta_{\xi^2}\tau(\mathbb{Z}_2^n) \) is trivial, we assume that \( \sigma_\xi \neq \sigma_{\xi^2} \). For \( i = 1,2 \), set \( C^{0,\xi^i} := \{ \alpha \in C \mid \langle \alpha, \xi^i \rangle = 0 \} \) and \( E := C^{0,\xi^1} \cap C^{0,\xi^2} \). Then \( C^{0,\xi^i} \) are subcodes of \( C \) with index 2 and \( E \) is a subcode with index 4. Let \( \delta^1, \delta^2 \in C \) such that \( C^{0,\xi^i} = E \cup (E + \delta^i) \). By definition, \( \alpha \cdot \xi^1, \alpha \cdot \xi^2 \in (E_0)^\perp \) for all \( \alpha \in D \) so that \( E_\alpha \) contains a self-dual subcode w.r.t. \( \alpha \) if and only if \( \langle \alpha \cdot \xi^1, \alpha \cdot \xi^2 \rangle = 0 \). We have seen that \( \theta_{\xi^i} \) acts semisimply on each \( V^\alpha \), \( \alpha \in D \), with two eigenvalues, and these eigenspaces are inequivalent irreducible \( V_{C^{0,\xi^i}} \)-submodules. For an irreducible \( V_E \)-submodule \( W \) of \( V^\alpha \), the subspace \( W + (V_E + \delta^i \cdot W) \) forms a \( V_{C^{0,\xi^i}} \)-submodule so that \( \theta_{\xi^i} \) acts on \( W \) by an eigenvalue. Therefore, \( \theta_{\xi^1} \) commutes with \( \theta_{\xi^2} \) if and only if \( V^\alpha \) splits into a direct sum of 4 irreducible \( V_E \)-submodules for all \( \alpha \in D \). Let \( m_\alpha \) be the number of irreducible \( V_E \)-submodules of \( V^\alpha \). For \( \alpha \in D \), let \( H_\alpha \) and \( H_0^\alpha \) be maximal self-orthogonal subcodes of \( C_\alpha \) and \( E_\alpha \), respectively. By the structure of irreducible modules over a code VOA shown in Theorem 3.12, \( V^\alpha \) is a direct sum of \( [C : H_\alpha] \) irreducible \( F \)-submodules. Moreover, again by Theorem 3.12 any irreducible \( V_E \)-submodule of \( V^\alpha \) is a direct sum of \( [E : H_0^\alpha] \) irreducible \( F \)-submodules. By counting the number of irreducible \( F \)-submodules of \( V^\alpha \), we have \( m_\alpha|H_\alpha| = 4|H_0^\alpha| \) as \( [C : E] = 4 \). Thus, \( E_\alpha \) contains a self-dual subcodes w.r.t. \( \alpha \) if and only if \( m_\alpha = 4 \). Hence, \( \theta_{\xi^1} \) commutes with \( \theta_{\xi^2} \) if and only if \( \langle \alpha \cdot \xi^1, \alpha \cdot \xi^2 \rangle = 0 \).

We have shown that the structure of \( \text{Stab}^\text{pt}_\tau(F) \) is determined by Theorems 3.50 and 3.54 only in terms of the structure codes \( (C, D) \).

**Remark 3.55.** In [Y2], [Y3], one of the authors has shown that for any Ising vector \( e \in V^3 \), we have no automorphism \( g \in \text{Aut}(V^2) \) such that \( g \) restricted on \( (V^2)^{(\tau_0)} \) is equal to \( \sigma_e \). Thanks to Theorem 3.50, we can give a simpler proof of this. For, take \( \xi = (10^{47}) \). Then there is no extension of \( \sigma_\xi = \sigma_{\xi^1} \) to \( \text{Stab}^\text{pt}_\tau(F) \) by Theorem 3.50. Since all the Ising vectors of \( V^3 \) are conjugate under \( \text{Aut}(V^2) = \mathfrak{M} \) (cf. [?, [M1]]), \( e \) and \( e^1 \) are conjugate. Therefore, there is no lift of \( \sigma_e \) on \( V^3 \).

At the end of this section, we give a brief description of the frame stabilizer \( \text{Stab}_F(F) \). Its structure is also discussed in [DGH]. It is clear that \( \text{Stab}^\text{pt}_\tau(F) \) is a normal subgroup of \( \text{Stab}_F(F) \). Let \( g \in \text{Stab}_F(F) \). Then \( g \) induces a permutation \( \mu_g \in S_n \) on the set of Ising vectors \( \{ e^1, \ldots, e^n \} \) of \( F \), namely \( g \cdot e^i = e^{\mu_g(i)} \). Since \( g \) preserves the 1/16-word decomposition \( V = \oplus_{\alpha \in D} V^\alpha \), it follows that \( gV^\alpha = V^\alpha \mu_g^{-1} \) with \( \mu_g^{-1}(\alpha) = (\alpha_\mu_g^{-1}(1), \ldots, \alpha_\mu_g^{-1}(n)) \). In particular, \( g \) restricted on \( V^0 \) defines an element of \( \text{Aut}(V^0) = \text{Aut}(V_C) \) which is a lift of
Aut(C). Therefore, every element of Stab_V(F) is a lift of Aut(C) ∩ Aut(D). Conversely, we know that for any μ ∈ Aut(C), there exists μ̄ ∈ Aut(V_C) such that μ̄e^i = e^{μ(i)} for 1 ≤ i ≤ n by Theorem 3.3 of [Sh]. It is shown in Lemma 3.15 of [SY] that if μ̄ lifts to an element of Aut(V) then \{(V^α)^{μ̄} | α ∈ D\} coincides with \{V^α | α ∈ D\} as a set of inequivalent irreducible V_C-modules. Therefore, there exists a lift of μ̄ ∈ Aut(V_C) = Aut(V^0) to an element of Aut(V) if and only if the subgroup \{V^α | α ∈ D\} of the group formed by all the simple current V_C-module in the fusion algebra is invariant under the conjugation action of μ̄. And such a lift of μ̄ is unique up to Stab_V(F) if it exists. Thus, the factor group Stab_V(F)/Stab_V^pt(F) is isomorphic to a subgroup of Aut(C) ∩ Aut(D) which states a slight refinement of (3) of Theorem 2.8 of [DGH]. Since the V_C-module structure (V^α)^{μ̄} requires some extra parameters other than C and D, we do not have a general result for the lifting property of Aut(C) ∩ Aut(D) at present.

### 3.6 4A-twisted orbifold construction

Let V^2 be the moonshine VOA constructed in [FLM]. In this section, we shall apply Theorem [3.50] to define a 4A-element of the Monster M = Aut(V^2) and exhibit that the 4A-twisted orbifold construction of the moonshine VOA V^2 will define V^2 itself.

It is shown in [DGH, M3] that V^2 has an Ising frame F = Vir(e^1) ⊗ · · · ⊗ Vir(e^{48}) such that the associated structure codes (C, D) are as follows:

\[ C = D^⊥, \quad D = \text{Span}_{Z_2}\{(11^4032), (0321^40), (α, α, α) | α ∈ RM(1, 4)\}, \]

where \( RM(1, 4) ⊂ Z_2^{48} \) is the first order Reed-Muller code defined by the generator matrix

\[
\begin{bmatrix}
1111 & 1111 & 1111 & 1111 \\
1111 & 1111 & 0000 & 0000 \\
1111 & 0000 & 1111 & 0000 \\
1100 & 1100 & 1100 & 1100 \\
1010 & 1010 & 1010 & 1010
\end{bmatrix}.
\]

Note that

\[ C = \{(α, β, γ) ∈ Z_2^{48} | α, β, γ ∈ Z_2^{16} \text{ are even and } α + β + γ ∈ RM(2, 4)\}. \quad (3.19) \]

**Remark 3.56.** The weight enumerator of RM(1, 4) is \( X^{16} + 30X^8Y^8 + Y^{16} \).

Let \( V^2 = \oplus_{α ∈ D}(V^2)^α \) be the 1/16-word decomposition. Set

\[ \mathcal{P} := \{ξ ∈ Z_2^{48} | α · ξ ∈ C \text{ for all } α ∈ D\}. \]

Then for each ξ ∈ \( \mathcal{P} \), one can define an automorphism θ_ξ ∈ Stab_V^pt(F) such that \( θ_ξ|_{V^0} = σ_ξ \) by Theorem [3.50]. Note also that \( D = C^⊥ < \mathcal{P} < C \) and \( σ_{ξ^1} = σ_{ξ^2} \) if and only if \( ξ^1 + ξ^2 ∈ C^⊥ = D \).

**Lemma 3.57.** Let C, D and P be defined as above. Then

\[ \mathcal{P} = \{(α, β, γ) ∈ Z_2^{48} | α, β, γ ∈ RM(2, 4) \text{ and } α + β + γ ∈ RM(1, 4)\}, \]

where \( RM(2, 4) = RM(1, 4)^⊥ ⊂ Z_2^{16} \) is the second order Reed-Muller code of length 16.
Proof: Set $E := \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \text{RM}(2, 4) \text{ and } \alpha + \beta + \gamma \in \text{RM}(1, 4)\}$. We shall first show that $E \subseteq P$. It is clear that if $d_1 \cdot \xi, d_2 \cdot \xi \in C$ then $(d_1 + d_2) \cdot \xi = d_1 \cdot \xi + d_2 \cdot \xi \in C$. Thus, we only need to show that $d \cdot \xi \in C$ for $\xi \in E$ and the generators of $D$.

Let $\xi = (\xi^1, \xi^2, \xi^3) \in E$ with each $\xi^i \in \mathbb{Z}_2^{16}$. Then $\xi^1, \xi^2, \xi^3 \in \text{RM}(4, 2)$. Hence, $\alpha \cdot \xi \in C$ for $\alpha = (1^{16}0^{32}), (0^{16}1^{16}0^{16})$ and $(0^{32}1^{16}) \in D$. Take any two codewords $\beta, \gamma \in \text{RM}(1, 4)$ with weight 8. Then the weight of $\beta \cdot \gamma$ is either 0, 4 or 8 by Remark 3.56. By the definition of $E$, we have $\xi^1 + \xi^2 + \xi^3 \in \text{RM}(1, 4)$. Therefore,

$$\langle \beta, \gamma \cdot (\xi^1 + \xi^2 + \xi^3) \rangle \equiv \text{wt}(\beta \cdot \gamma \cdot (\xi^1 + \xi^2 + \xi^3)) \equiv 0 \mod 2$$

and hence $\gamma \cdot (\xi^1 + \xi^2 + \xi^3) \in \text{RM}(1, 4) \subseteq \text{RM}(2, 4)$. Since $\xi^i \in \text{RM}(2, 4) = \text{RM}(1, 4) \subseteq \text{RM}(2, 4)$, all $\gamma \cdot \xi^i, i = 1, 2, 3$, are even codewords. Therefore, $(\beta, \gamma, \gamma) \cdot \xi = (\gamma \cdot \xi^1, \gamma \cdot \xi^2, \gamma \cdot \xi^3) \in C$ for all $\gamma \in \text{RM}(1, 4)$. As $D$ is generated by the elements of the form: $(1^{16}0^{32}), (0^{16}1^{16}0^{16}), (0^{32}1^{16})$ and $(\gamma, \gamma, \gamma)$ with $\gamma \in \text{RM}(1, 4)$, we have $E \subseteq P$.

Conversely, assume $(\alpha^1, \alpha^2, \alpha^3) \in P$ with $\alpha^i \in \mathbb{Z}_2^{16}$. Then one has $(\alpha^1, \alpha^2, \alpha^3) \cdot \beta \in C$ for $\beta = (1^{16}0^{32}), (0^{16}1^{16}0^{16})$ and $(0^{32}1^{16}) \in D$ so that $\alpha^i \in \text{RM}(2, 4)$ for $i = 1, 2, 3$. Moreover, for $(\gamma, \gamma, \gamma) \in D$ with $\gamma \in \text{RM}(1, 4)$, $(\alpha^1, \alpha^2, \alpha^3) \cdot (\gamma, \gamma, \gamma) \in C$ is an even codeword. Then it follows from (3.19) that $(\alpha^1 + \alpha^2 + \alpha^3) \cdot \gamma \in \text{RM}(2, 4)$ and thus $\alpha^1 + \alpha^2 + \alpha^3 \in \text{RM}(1, 4)$. Hence $E = P$. 

Take

$$\xi = (1100 \ 0000 \ 1100 \ 0000 \ 0110 \ 0000 \ 0110 \ 0000 \ 1010 \ 0000 \ 1010 \ 0000) \in \mathbb{Z}_2^{48}. \quad (3.20)$$

Then $\xi \in P$ by Lemma 3.57. Set $D^0 = \{\alpha \in D \mid \text{wt}(\alpha \cdot \beta) \equiv 0 \mod 4\}$ and $D^1 = \{\alpha \in D \mid \text{wt}(\alpha \cdot \beta) \equiv 2 \mod 2\}$. It is easy to see that

$$D^0 = \text{Span}_{\mathbb{Z}_2}\{(1^{16}0^{32}), (0^{32}1^{16}), (\alpha, \alpha, \alpha) \mid \alpha = (1^{16}), (1^40^4)^2, (1^20^2)^4\}$$

and $D^1 = (1^{16}0^{32}) + D^0$. Therefore, the index $[D : D^0]$ is 2 and in this case an involution $\sigma_\xi \in \text{Aut}(V^2)$ can be extended to a pointwise frame stabilizer $\theta_\xi \in \text{Stab}^F_{V^2}(F)$ of order 4 by Theorem 3.50. We also set $C^0 := \{\alpha \in C \mid \langle \alpha, \xi \rangle = 0\}$ and $C^1 := \{\alpha \in C \mid \langle \alpha, \xi \rangle = 1\}$. Let us consider a subgroup of $D \times C^*$ defined by

$$\bar{D} := (D^0 \times \{\pm 1\}) \sqcup (D^1 \times \{\pm \sqrt{-1}\}). \quad (3.21)$$

For $(\alpha, u) \in D \times C^*$, set $(V^2)^{(\alpha, u)} := \{x \in (V^2)^u \mid \theta_\xi x = ux\}$. Then we have a $\bar{D}$-graded decomposition

$$V^2 = \bigoplus_{(\alpha, u) \in \bar{D}} (V^2)^{(\alpha, u)}, \quad (V^2)^{(0, 1)} = V_{C^0}, \quad (V^2)^{(0, -1)} = V_{C^1}. \quad (3.22)$$

where $V_{C^0}$ denotes the code VOA associated to $C^0$ and $V_{C^1}$ is its module. Since $(C^0)_\alpha$ contains a self-dual subcode w.r.t. $\alpha \in \bar{D}$ by Lemma 3.48, all $(V^2)^{(\alpha, u)}, (\alpha, u) \in \bar{D}$, are simple current $V_{C^0}$-modules by Corollary 3.25. Therefore, $V^2$ is a $\bar{D}$-graded simple current extension of a code VOA $(V^2)^{(0, 1)} = V_{C^0}$.

By direct computation, it is not difficult to obtain the following lemma.
Lemma 3.58. For any non-zero $\alpha \in D^0$, the subset $(C^1)_\alpha$ is not empty. In other words, $[C_\alpha, (C^0)_\alpha] = 2$.

Remark 3.59. For a general framed VOA with structure codes $(C, D)$, it is possible that the set $(C^1)_\alpha$ is empty for some non-zero $\alpha \in D$. Indeed there exists such an example.

Theorem 3.60. $\theta_\xi$ is a $4A$-element of the Monster.

Proof: We shall compute the McKay-Thompson series of $\theta_\xi$.

$$ T_{\theta_\xi}(z) = \text{tr}_{V_\xi} \theta_\xi q^{L(0) - 1}, \quad q = e^{2\pi \sqrt{-1} z}. $$

Recall the notion of the conformal character of a module $M = \bigoplus_{n \geq 0} M_{n+h}$ over a VOA $V$:

$$ \text{ch}_M(z) = \text{tr}_M q^{L(0) - c/24} = \sum_{n=0}^{\infty} \dim_C M_{n+h} q^{n+h-c/24}. $$

It is clear that

$$ T_{\theta_\xi}(z) = \sum_{\alpha \in D^0} \text{ch}_{(V^2)^{(\alpha,1)}}(z) - \sum_{\alpha \in D^0} \text{ch}_{(V^2)^{(\alpha,-1)}}(z) $$

$$ + \sqrt{-1} \sum_{\alpha \in D^1} \text{ch}_{(V^2)^{(\alpha,\pm \sqrt{-1})}}(z) - \sqrt{-1} \sum_{\alpha \in D^1} \text{ch}_{(V^2)^{(\alpha,-\sqrt{-1})}}(z). $$

Let $\alpha \in D^1$. Since $(V^2)^{(\alpha,\pm \sqrt{-1})}$ are dual to each other, their conformal characters are the same. Let $\alpha \in D^0$ be a non-zero codeword. By Lemma 3.58, there exists a code word in $(C^1)_\alpha$. Then by Corollary 3.16, one sees that $(V^2)^{(\alpha,1)}$ and $(V^2)^{(\alpha,-1)}$ are isomorphic $F$-modules. Therefore, they have also the same conformal characters. Then

$$ T_{\theta_\xi}(z) = \text{ch}_{(V^2)^{(\alpha,1)}}(z) - \text{ch}_{(V^2)^{(\alpha,-1)}}(z) = \text{ch}_{C^{\alpha}}(z) - \text{ch}_{C^{\alpha}}(z) = 2\text{ch}_{C^{\alpha}}(z) - \text{ch}_{C}(z). $$

The conformal character of a code VOA can be easily computed. The following conformal characters are well-known (cf. [FFR]):

$$ \text{ch}_{L(1/2,0)}(z) = h_{1/2,0}(z) = q^{-1/48} \prod_{n=0}^{\infty} (1 \pm q^{n+1/2}). $$

Since $C = D^\perp$ and $C^0 = (D + \langle \xi \rangle)^\perp$, the weight enumerators of these codes are calculated by the MacWilliams identity [McS]:

$$ W_C(x, y) = \frac{1}{|D|} W_D(x+y, x-y), $$

$$ W_{C^0}(x, y) = \frac{1}{|D + \langle \xi \rangle|} W_D(x+y, x-y), $$

where

$$ W_D(x, y) = x^{48} + 3x^{32}y^{16} + 120x^{24}y^{24} + 3x^{16}y^{32} + y^{48}, $$

$$ W_{D+\langle \xi \rangle}(x, y) = x^{48} + 2x^{36}y^{12} + 3x^{32}y^{16} + 30x^{28}y^{20} + 184x^{24}y^{24} + 30x^{20}y^{28} + 3x^{16}y^{32} + 2x^{12}y^{36} + y^{48}. $$

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Now set
\[ f(x, y) := W_{D+\xi}(x, y) - W_{D}(x, y) = 2x^{36}y^{12} + 30x^{28}y^{20} + 64x^{24}y^{24} + 30x^{20}y^{28} + 2x^{12}y^{36}. \]

Then one has
\[ T_{\theta_\xi}(z) = 2\text{ch}_{V_{C_0}}(z) - \text{ch}_{V_C}(z) = [2W_{C_0}(x, y) - W_C(x, y)]_{x=\text{ch}_{L(1/2,0)}(z), y=\text{ch}_{L(1/2,1)}(z)} = 1/27 [W_{D+(\xi)}(x+y, x-y) - W_{D}(x+y, x-y)]_{x=\text{ch}_{L(1/2,0)}(z), y=\text{ch}_{L(1/2,1)}(z)} = 1/27 q^{-1}f \left( \prod_{n=0}^{\infty} (1 + q^{n+1/2}), \prod_{n=0}^{\infty} (1 - q^{n+1/2}) \right) = q^{-1} + 276q + 2048q^2 + \cdots. \]

Therefore, \( \theta_\xi \) is a 4A-element of the Monster by [ATLAS].

Next, we shall construct the irreducible 4A-twisted \( V^3 \)-module. Let us consider an irreducible \( V_{C_0} \)-module \( W = M_{C_0}(\xi, 0) \), where \( \xi \) is defined as in (3.20).

**Lemma 3.61.** (1) \( (C^0)_\xi \) is a self-dual subcode w.r.t. \( \xi \).
(2) The dual \( V_{C_0} \)-module of \( W \) is isomorphic to \( M_{C_0}(\xi, \kappa) \) with \( \kappa = \{10^7\}^2 0^{32} \in \mathbb{Z}_2^{48} \).

**Proof:** By a direct computation, one can show that \( C_\xi = (C^0)_\xi \) is generated by the following generator matrix:

\[
\begin{bmatrix}
11000000 & 11000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \\
00000000 & 00000000 & 01100000 & 01100000 & 00000000 & 00000000 & 00000000 \\
00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 10100000 & 10100000 \\
10000000 & 10000000 & 01000000 & 01000000 & 00000000 & 00000000 & 00000000 \\
10000000 & 10000000 & 00000000 & 00000000 & 10000000 & 10000000 & 00000000 \\
11000000 & 00000000 & 01100000 & 00000000 & 00000000 & 01000000 & 00000000 \\
\end{bmatrix}
\]

From this, it is easy to see that \( (C^0)_\xi \) is a self-dual code w.r.t. \( \xi \). If we set
\[
\kappa = (10000000 10000000 00000000 00000000 00000000 00000000 00000000) \in \mathbb{Z}_2^{48},
\]
then \( \text{supp}(\kappa) \subset \text{supp}(\xi) \) and we have \((-1)^{(\alpha, \kappa)} = (-1)^{\text{wt}(\alpha)/2} \) for all \( \alpha \in (C^0)_\xi \). Therefore, the dual \( W^* \) of \( W \) is isomorphic to \( M_{C_0}(\xi, \kappa) \) by Proposition 3.18.

**Lemma 3.62.** All \( (V^2)^{(\alpha,u)} \otimes_{V_{C_0}} W \), \( (\alpha, u) \in \bar{\mathcal{D}} \), are inequivalent irreducible \( V_{C_0} \)-modules.

**Proof:** Suppose \((V^2)^{(\alpha,u)} \otimes_{V_{C_0}} W \simeq (V^2)^{(\beta,v)} \otimes_{V_{C_0}} W \) with \((\alpha, u), (\beta, v) \in \bar{\mathcal{D}} \). Since \(((V^2)^{(\alpha,u)})^* \simeq (V^2)^{(\alpha,u)^{-1}} \simeq (V^2)^{(\alpha+\beta,u^{-1})} \), we have
\[
W = (V^2)^{(\alpha,u^{-1})} \otimes_{V_{C_0}} (V^2)^{(\alpha,u)} \otimes_{V_{C_0}} W
= (V^2)^{(\alpha,u^{-1})} \otimes_{V_{C_0}} (V^2)^{(\beta,v)} \otimes_{V_{C_0}} W
= (V^2)^{(\alpha+\beta,u^{-1})} \otimes_{V_{C_0}} W
\]

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in the fusion algebra. By considering 1/16-word decompositions, one has \( \alpha = \beta \) and \( u^{-1}v \in \{ \pm 1 \} \). Let \( \delta \in \mathcal{C} \) be such that \( \mathcal{C}^1 = \mathcal{C}^0 + \delta \). If \( u^{-1}v = -1 \), then \( (V^z)^{(\alpha + \beta, u^{-1}v)} = (V^z)^{(0, -1)} = V_{\mathcal{C}^1} = M_{\mathcal{C}^0}(0, \delta) \) so that by Lemma 3.21 one has

\[
W = (V^z)^{(\alpha + \beta, u^{-1}v)} \boxtimes W = M_{\mathcal{C}^0}(0, \delta) \boxtimes M_{\mathcal{C}^0}(\xi, 0) = M_{\mathcal{C}^0}(\xi, \delta).
\]

It is shown in Lemma 3.61 that \( (\mathcal{C}^0)_\xi \) contains a self-dual subcode w.r.t. \( \xi \). Then \( W \) is not isomorphic to \( M_{\mathcal{C}^0}(\xi, \delta) \) by Lemma 3.15 and we obtain a contradiction. Therefore, \( u^{-1}v = 1 \) and hence \((\alpha, u) = (\beta, v)\). This completes the proof.

By Theorem 3.4 and the lemma above, the space

\[
V^z(\theta_\xi) := V^z \boxtimes W = \bigoplus_{(\alpha, u) \in \mathcal{D}} (V^z)^{(\alpha, u)} \boxtimes M_{\mathcal{C}^0}(\xi, 0)
\]

(3.24)
carries a unique structure of an irreducible \( \chi_W \)-twisted \( V^z \)-module for some \( \chi_W \in (\mathcal{D})^* \subset \mathbb{M} \). It is clear that \( \chi_W \in \text{Stab}_{V^z}(F) \). Since the top weight of \( W \) is \( 3/4 \), \( V^z(\theta_\xi) \) is neither 2A-twisted nor 2B-twisted \( V^z \)-module so that \( \chi_W \) is a 4A-element of \( \mathbb{M} \) by Theorem 3.60. Hence, there exists \( \xi' \in \mathcal{P} \) such that \( \chi_W(\nu_{V^z}) = \sigma_{\xi'} \). By the construction of \( V^z(\theta_\xi) \), we know that \( \mathcal{C}^0 \subset \{ \alpha \in \mathcal{C} \mid \langle \alpha, \xi' \rangle = 0 \} \subset \mathcal{C} \) so that \( \sigma_\xi = \sigma_{\xi'} \) and \( \chi_W = \theta_\xi \cdot \tau_\eta \) for some \( \eta \in \mathbb{Z}_2^{48} \). Since the definition of \( \theta_\xi \) is only unique up to a product of \( \tau \)-involutions, we can choose \( \theta_\xi \) to be equal to \( \chi_W \). Note that the definition of \( V^z(\theta_\xi) \) in (3.24) depends only on the decomposition of \( V^z \) in (3.22) as a \( V_{\mathcal{C}^0} \)-module. Replacing \( \theta_\xi \) by \( \theta_\xi \tau_\beta \) with \( \beta \in \mathbb{Z}_2^{48} \) will only change the labeling of the \( V_{\mathcal{C}^0} \)-modules in (3.22) and does not affect the isotypical decomposition of \( V^z \) as a \( V_{\mathcal{C}^0} \)-module. Thus, we have constructed the irreducible \( \theta_\xi \)-twisted \( V^z \)-module.

**Theorem 3.63.** \( V^z(\theta_\xi) \) defined in (3.24) is an irreducible 4A-twisted module over \( V^z \).

**Remark 3.64.** By Lemma 3.61 and Corollary 3.16 we see that the top weight of the 4A-twisted module is \( 3/4 \) and the dimension of the top level is \( 1 \).

Let us consider the dual module \( W^* \) of \( W \). It is clear that \( W \) and \( W^* \) are inequivalent since \( \kappa \notin \mathcal{C}^0 \). All \( (V^z)^{(\alpha, u)} \boxtimes W^* \), \((\alpha, u) \in \mathcal{D} \), are inequivalent \( V_{\mathcal{C}^0} \)-modules, and so the space

\[
V^z(\theta^3_\xi) := V^z \boxtimes W^* = \bigoplus_{(\alpha, u) \in \mathcal{D}} (V^z)^{(\alpha, u)} \boxtimes M_{\mathcal{C}^0}(\xi, \kappa)
\]

(3.25)
uniquely forms an irreducible \( \chi_W \)-twisted \( V^z \)-module for a linear character \( \chi_W \in (\mathcal{D})^* \) by Theorem 3.4. It is shown in [DLM2] that the dual of \( \chi_W \)-twisted module forms a \( \chi_W^{-1} \)-twisted module. The dual \( V^z \)-module of \( V^z(\theta_\xi) \) contains \( W^* \) as a \( V_{\mathcal{C}^0} \)-submodule, and \( V^z(\theta^3_\xi) \) is uniquely determined by \( W^* \), so the character \( \chi_W \)- is actually equal to \( \theta^3_\xi = \theta^{-1}_\xi \) by our choice of \( \theta_\xi \). Therefore, \( V^z(\theta^3_\xi) \) is the irreducible \( \theta^3_\xi \)-twisted \( V^z \)-module.

In order to perform the 4A-twisted orbifold construction of \( V^z \), we classify the irreducible representations of \( (V^z)^{(\theta_\xi)} \). By (3.22), the fixed point subalgebra \( (V^z)^{(\theta_\xi)} \) is a framed VOA with structure codes \((\mathcal{C}^0, \mathcal{D}^0)\).
Proposition 3.56. There are 16 inequivalent irreducible $(V^2)^{(θ_κ)}$-modules. Every irreducible $(V^2)^{(θ_κ)}$-module is a submodule of an irreducible $θ_κ^i$-twisted $V^2$ for $0 ≤ i ≤ 3$. Among them, 8 irreducible modules have integral top weights.

Proof: Since $V^2$ and $(V^2)^{(θ_κ)}$ are simple current extensions of the code VOA $V_{C^0}$, $V^2$ is a $Z_4$-graded simple current extension of $(V^2)^{(θ_κ)}$ by Proposition 3.3. Then by Theorem 3.4, every irreducible $(V^2)^{(θ_κ)}$-module is a submodule of a $θ_κ^i$-twisted module for $0 ≤ i ≤ 3$. It is shown in [DLM3] that $V^2$ has a unique irreducible $θ_κ^i$-twisted module for $0 ≤ i ≤ 3$ as $V^2$ is holomorphic. Therefore, we only have to show that each irreducible $θ_κ^i$-twisted $V^2$-module decomposes into a direct sum of 4 inequivalent irreducible $(V^2)^{(θ_κ)}$-submodules. Since $V^2(θ_κ)$ and $V^2(θ_κ^2)$ are dual to each other (cf. [DLM2]), we know each of them has four inequivalent irreducible $(V^2)^{(θ_κ)}$-submodules. So we consider a $θ_κ^2$-twisted module. Let $κ ∈ Z_2$ be as in (3.23). Then one can easily verify that $θ_κ^2 = τ_κ$. We consider an irreducible $V_{C^0}$-module $M_{C^0}(0, κ)$. We claim that $(V^2)^{(α,u)}(α) ⊗ V_{C^0} M_{C^0}(0, κ), (α, u) ∈ D$, are inequivalent irreducible $V_{C^0}$-modules. The irreducibility of $(V^2)^{(α,u)} ⊗ V_{C^0} M_{C^0}(0, κ)$ is clear since all $(V^2)^{(α,u)}$ are simple current $V_{C^0}$-modules. If $(V^2)^{(α,u)} ⊗ V_{C^0} M_{C^0}(0, κ) ≃ (V^2)^{(β,v)} ⊗ V_{C^0} M_{C^0}(0, κ)$, then by the 1/16-word decomposition, we have $α = β$ and $u^{-1}v ∈ \{±1\}$. Let $δ ∈ Z_2$ such that $C₁ = C^0 + δ$. If $u^{-1}v = −1$, then $(V^2)^{(α+β,u^{-1}v)} = M_{C^0}(0, δ)$ and one has

$$M_{C^0}(0, κ) = (V^2)^{(α,u^{-1})} ⊗ V_{C^0} (V^2)^{(β,v)} ⊗ V_{C^0} M_{C^0}(0, κ)$$

But this is a contradiction by Lemma 3.15. Therefore, all $(V^2)^{(α,u)} ⊗ V_{C^0} M_{C^0}(0, κ), (α, u) ∈ D$, are inequivalent irreducible $V_{C^0}$-modules. Now by Theorem 3.4 and Lemma 3.21, the space

$$V^2(θ_κ^2) := \bigoplus_{(α,u) ∈ D} (V^2)^{(α,u)} ⊗ V_{C^0} M_{C^0}(0, κ) \tag{3.26}$$

forms an irreducible $τ_κ = θ_κ^2$-twisted $V^2$-module. Thus $V^2(θ_κ^2)$ splits into four irreducible $(V^2)^{(θ_κ)}$-submodules as follows.

$$V^2(θ_κ^2) = \bigoplus_{α ∈ D₀} (V^2)^{(α,1)} ⊗ V_{C^0} M_{C^0}(0, κ) \bigoplus_{α ∈ D₀} (V^2)^{(α,-1)} ⊗ V_{C^0} M_{C^0}(0, κ)$$

$$\bigoplus_{α ∈ D₁} (V^2)^{(α,−1)} ⊗ V_{C^0} M_{C^0}(0, κ) \bigoplus_{α ∈ D₁} (V^2)^{(α,−1)} ⊗ V_{C^0} M_{C^0}(0, κ)$$

Therefore, all irreducible $θ_κ^i$-twisted $V^2$-modules are direct sums of 4 inequivalent irreducible $(V^2)^{(θ_κ)}$-submodules and we have obtained 16 irreducible $(V^2)^{(θ_κ)}$-modules. It remains to show that these 16 modules are inequivalent. Since every irreducible $(V^2)^{(θ_κ)}$-module can be uniquely extended to a $θ_κ$-twisted $V^2$-module by Theorem 3.4, these 16 irreducible modules are actually inequivalent.
Among these 16 irreducible \((V^2)^{(\theta_4)}\)-modules, we have 4 modules having integral top weight from \(V^2\), 2 from \(\theta_4^2\)-twisted \(V^2\)-module, 1 from \(\theta_4\)-twisted and 1 from \(\theta_4^2\)-twisted modules, respectively. This completes the proof.

We have also shown that every irreducible \(\theta_4^i\)-twisted \(V^2\)-module has a \(\mathbb{Z}_4\)-grading which agrees with the action of \(\theta_4\) on \(V^2\). By this fact, we adopt the following notation.

For \(u \in \mathbb{C}^*\) satisfying \(u^4 = 1\), we set \(V^2(1, u) := \{ a \in V^2 \mid \theta_4 a = u a \}\). For \(i = 1\) or 3, we define \(V^2(\theta_4^i, 1)\) to be the unique irreducible \((V^2)^{(\theta_4^i)}\)-submodule of \(V^2(\theta_4^i)\) which has integral top weight. They can be defined explicitly as follows.

\[
V^2(\theta_4^i, 1) := \bigoplus_{\alpha \in \mathcal{D}} (V^2)^{(\alpha, -\sqrt{-1})} \boxtimes_{V_{c_0}} M_{c_0}(\xi, 0),
\]

\[
V^2(\theta_4^2, 1) := \bigoplus_{\alpha \in \mathcal{D}} (V^2)^{(\alpha, \sqrt{-1})} \boxtimes_{V_{c_0}} M_{c_0}(\xi, \kappa).
\]

For \(i = 2\), there are two irreducible \((V^2)^{(\theta_4^i)}\)-submodules in \(V^2(\theta_4^2)\) having integral top weights. We shall define

\[
V^2(\theta_4^2, 1) := \bigoplus_{\alpha \in \mathcal{D}} (V^2)^{(\alpha, -1)} \boxtimes_{V_{c_0}} M_{c_0}(0, \kappa),
\]

\[
V^2(\theta_4^2, -1) := \bigoplus_{\alpha \in \mathcal{D}} (V^2)^{(\alpha, 1)} \boxtimes_{V_{c_0}} M_{c_0}(0, \kappa).
\]

In addition, we define

\[
V^2(\theta_4^i, u) := V^2(1, u) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(\theta_4^i, 1) \quad \text{for} \quad 1 \leq i \leq 3.
\]

Set \(G := \langle \theta \rangle \times \{ u \in \mathbb{C}^* \mid u^4 = 1 \}\). Then \(\{ V^2(g, u) \mid (g, u) \in G \}\) is the set of all inequivalent irreducible \((V^2)^{(\theta_4^i)}\)-modules.

**Proposition 3.66.** The fusion algebra associated to \((V^2)^{(\theta_4^i)}\) is isomorphic to the group algebra of \(G\). The isomorphism is given by \(V^2(g, u) \mapsto (g, u)\).

**Proof:** Since the structure codes of \((V^2)^{(\theta_4^i)}\) is \((C_0, D^0)\), \((V^2)^{(\theta_4^i)}\) is a \(D^0\)-graded simple current extension of \(V_{c_0}\). So we have the following fusion rules:

\[
V^2(1, u) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(1, v) = V^2(1, uv) \quad \text{for} \quad u, v \in \mathbb{C}^*, \quad u^4 = v^4 = 1.
\]

Since all \(V^2(g, u), (g, u) \in G\), are \(D^0\)-stable, we can use Proposition 3.5. The following fusion rules of \(V_{c_0}\)-modules are already known:

\[
M_{c_0}(0, \kappa) \boxtimes_{V_{c_0}} M_{c_0}(0, \kappa) = M_{c_0}(0, 0), \quad M_{c_0}(0, \kappa) \boxtimes_{V_{c_0}} M_{c_0}(\xi, 0) = M_{c_0}(\xi, \kappa),
\]

\[
M_{c_0}(\xi, 0) \boxtimes_{V_{c_0}} M_{c_0}(\xi, 0) = M_{c_0}(0, \kappa), \quad M_{c_0}(\xi, 0) \boxtimes_{V_{c_0}} M_{c_0}(\xi, \kappa) = M_{c_0}(0, 0).
\]

Therefore, we have the following fusion rules of \((V^2)^{(\theta_4^i)}\)-modules:

\[
V^2(\theta_4^2, 1) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(\theta_4^2, 1) = V^2(1, 1), \quad V^2(\theta_4^2, 1) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(\theta_4^2, 1) = V^2(\theta_4^2, 1),
\]

\[
V^2(\theta_4^2, 1) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(\theta_4^2, 1) = V^2(\theta_4^2, 1), \quad V^2(\theta_4^2, 1) \boxtimes_{(V^2)^{(\theta_4^i)}} V^2(\theta_4^2, 1) = V^2(1, 1).
\]
Since the fusion algebra is commutative and associative, the remaining fusion rules are deduced from the above and we can establish the isomorphism.

A \( \theta_{\xi} \)-twisted orbifold construction of \( V^z \) refers to a construction of a \( \mathbb{Z}_4 \)-graded (simple current) extension of the \( \theta_{\xi} \)-fixed point subalgebra \( (V^z)^{\theta_{\xi}} \) by using the irreducible submodules of \( V^z(\theta_{\xi}^2) \) with integral weights. By Proposition 3.65 such modules are denoted by

\[
V^z(1, \pm 1), \quad V^z(1, \pm \sqrt{-1}), \quad V^z(\theta_{\xi}^2, \pm 1), \quad V^z(\theta_{\xi}, 1) \quad \text{and} \quad V^z(\theta_{\xi}^3, 1).
\]

By the fusion rules in Proposition 3.66 there are three possible extensions of \( (V^z)^{\theta_{\xi}} \), namely,

\[
\begin{align*}
V^z &= V^z(1, 1) \oplus V^z(1, -1) \oplus V^z(1, \sqrt{-1}) \oplus V^z(1, -\sqrt{-1}), \\
V_{2B} &= V^z(1, 1) \oplus V^z(1, -1) \oplus V^z(\theta_{\xi}^2, 1) \oplus V^z(\theta_{\xi}, -1), \\
V_{4A} &= V^z(1, 1) \oplus V^z(\theta_{\xi}, 1) \oplus V^z(\theta_{\xi}^2, 1) \oplus V(\theta_{\xi}^3, 1).
\end{align*}
\]

By the original construction of the moonshine VOA in \( \text{FLM} \), the \( \theta_{\xi}^2 \)-fixed point subalgebra \( (V^z)^{\theta_{\xi}^2} = V^z(1, 1) \oplus V^z(1, -1) \) is isomorphic to the fixed point subalgebra \( V^z_\Lambda \) of the Leech lattice VOA \( V_\Lambda \). It is shown in \( \text{DL} \) that \( V^z_\Lambda \) has four inequivalent irreducible modules which are denoted by \( V^z_\Lambda^T \) and \( V^z_\Lambda^T \) in \( \text{FLM} \). Since the top weights of irreducible \( V^z_\Lambda^+ \)-modules belong to \( \mathbb{Z}/2 \), the inequivalent irreducible \( (V^z)^{\theta_{\xi}} \)-modules are given by the list below:

\[
\begin{align*}
V^z(1, 1) \oplus V^z(1, -1), \quad V^z(1, \sqrt{-1}) \oplus V^z(1, -\sqrt{-1}), \\
V^z(\theta_{\xi}^2, 1) \oplus V^z(\theta_{\xi}, -1), \quad V^z(\theta_{\xi}^2, \sqrt{-1}) \oplus V^z(\theta_{\xi}, -\sqrt{-1}).
\end{align*}
\]

It is shown in \( \text{H} \) that there are two inequivalent simple extensions of \( V^z_\Lambda^+ \): one is the moonshine VOA \( V^z \) and the other is \( V_\Lambda \). We prove that \( \theta_{\xi}^2 \)-twisted orbifold construction \( V_{2B} \) in (3.29) is isomorphic to the Leech lattice VOA \( V_\Lambda \). For this, it is enough to show the following.

**Lemma 3.67.** The top weight of \( V^z(\theta_{\xi}^2, -1) \) is 1 and the dimension of the top level is 24.

**Proof:** Recall the 1/16-word decomposition (3.27) of \( V^z(\theta_{\xi}^2, -1) \) as a module over \( V_{C^0} \). It contains a \( V_{C^0} \)-submodule

\[
(V^z)^{(0,1)} \otimes_{V_{C^0}} M_{C^0}(0, \kappa) = M_{C^0}(0, \kappa) = V_{C^0 + \kappa}.
\]

By a straightforward computation we see that there are exactly 24 weight two codewords in the coset \( C^0 + \kappa \) and the support of each is one of the following:

\[
\begin{align*}
\{1, 9\}, \quad \{2, 10\}, \quad \{3, 11\}, \quad \{4, 12\}, \quad \{5, 13\}, \quad \{6, 14\}, \quad \{7, 15\}, \quad \{8, 16\}, \\
\{17, 25\}, \quad \{18, 26\}, \quad \{19, 27\}, \quad \{20, 28\}, \quad \{21, 29\}, \quad \{22, 30\}, \quad \{23, 31\}, \quad \{24, 32\}, \\
\{33, 41\}, \quad \{34, 42\}, \quad \{35, 43\}, \quad \{36, 44\}, \quad \{37, 45\}, \quad \{38, 46\}, \quad \{39, 47\}, \quad \{40, 48\}.
\end{align*}
\]

Therefore, the top weight of \( V^z(\theta_{\xi}^2, -1) \) is 1. Then by the list of irreducible \( V^z_\Lambda^+ \)-modules in (3.30), the dimension of the top level of \( V^z(\theta_{\xi}^2, -1) \) must be 24. Or, one can directly check that all \( (V^z)^{(0,1)} \otimes_{V_{C^0}} M_{C^0}(0, \kappa) \) has the top weight greater than 1 for any non-zero \( \alpha \in D^0 \) by considering their \( F \)-module structures.
Thus, by considering top weights and fusion rules, we have the following isomorphisms:

\[ V^+_A \simeq V^2(1, 1) \oplus V^2(1, -1), \quad V^+_A \simeq V^2(1, \sqrt{-1}) \oplus V^2(1, -\sqrt{-1}), \]
\[ V^-_A \simeq V^2(\theta_\xi^2, 1) \oplus V^2(\theta_\xi^2, -1), \quad V^-_A \simeq V^2(\theta_\xi^2, \sqrt{-1}) \oplus V^2(\theta_\xi^2, -\sqrt{-1}). \]  \hfill (3.32)

Therefore, the \( \theta_\xi^2 \)-twisted orbifold construction \( V_{2B} \) is isomorphic to \( V_A \).

Remark 3.68. One can identify \( V_{2B} \) with the Leech lattice VOA without the isomorphisms (3.32). For, \( V_{2B} \) can be defined as a framed VOA with structure codes \((\mathcal{C}^0 \cup \mathcal{C}^1 \cup (\mathcal{C}^0 + \kappa) \cup (\mathcal{C}^1 + \kappa), \mathcal{D}^0)\) by Theorem 3.37 which is holomorphic by Corollary 3.36. It follows from (3.31) that \( V_{2B} \) contains a free bosonic VOA associated to a metric space of rank 24. Then \( V_{2B} \) is isomorphic to a lattice VOA associated to an even unimodular lattice by [LiX]. Since the weight one subspace of \( V_{2B} \) is 24-dimensional, \( V_{2B} \) is actually isomorphic to the lattice VOA associated to the Leech lattice. Similarly, one can also show that \((V^2)^{\theta_\xi^2} = V^2(1, 1)\) is isomorphic to a \( \mathbb{Z}_2 \)-orbifold \( V_L^+ \) of a lattice VOA \( V_L \) for certain sublattice \( L \) of \( A \). For, we know that \( V^2(1, 1) \oplus V^2(\theta_\xi^2, -1) \) forms a sub VOA of \( V_{2B} \) isomorphic to \( V_L \). Since \( V^2(1, 1) \) is a \( \mathbb{Z}_2 \)-fixed point subalgebra of \( V^2(1, 1) \oplus V^2(\theta_\xi^2, -1) \) under an involution acting on \( V^2(\theta_\xi^2, -1) \) by \(-1\), we have the isomorphism as claimed. This isomorphism is first pointed out by Shimakura in a different viewpoint.

Next we consider the proper \( \theta_\xi \)-twisted orbifold construction \( V_{4A} \) in (3.29). Let \( \alpha \in \mathcal{D}^1 \) be arbitrary. We can find the following \( V_{c0} \)-submodule in \( V_{4A} \).

\[ U := V_{c0} \oplus V_{c1+\kappa} \oplus \{(V^2)^{(\alpha, -\sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, 0)\} \oplus \{(V^2)^{(\alpha, \sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, \kappa)\}. \]

Lemma 3.69. There exists a unique structure of a framed VOA on \( U \).

Proof: Set \( U^0 = V_{c0} \oplus V_{c1+\kappa} \) and

\[ U^1 = \{(V^2)^{(\alpha, -\sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, 0)\} \oplus \{(V^2)^{(\alpha, \sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, \kappa)\}. \]

Then \( U^0 \) is a code VOA associated to \( \mathcal{C}^0 \cup (\mathcal{C}^1 + \kappa) \) and \( U^1 \) is an irreducible \( U^0 \)-module. By using Propositions 3.5 and 3.22 one can verify that \( U^1 \boxtimes U^0 \) is isomorphic to \( U^0 \) which shows that \( U^1 \) is a self-dual simple current \( U^0 \)-module. Therefore, by Proposition 3.6, \( U = U^0 \oplus U^1 \) forms a framed VOA if and only if the \( U^0 \)-invariant bilinear forms on \( U^1 \) is symmetric. Let \( \langle \cdot, \cdot \rangle_{U^1} \) be a \( U^0 \)-invariant bilinear form on \( U^1 \). It is shown in [Li1] that \( \langle \cdot, \cdot \rangle_{U^1} \) is either symmetric or sym-skew-symmetric. Take a non-zero highest weight vector \( u \in (V^2)^{(\alpha, \sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, 0) \). Since \( V_{c0} \)-modules \( (V^2)^{(\alpha, -\sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, 0) \) and \( (V^2)^{(\alpha, \sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, \kappa) \) are dual to each other, we can take a highest weight vector \( v \in (V^2)^{(\alpha, \sqrt{-1})} \boxtimes V_{c0} M_{c0}(\xi, \kappa) \) such that \( \langle u, v \rangle_{U^1} = \langle v, u \rangle_{U^1} = 0 \) and \( \langle u, v \rangle_{U^1} = 1 \). Then \( \langle u + v, u + v \rangle_{U^1} = 2 \neq 0 \) so that \( \langle \cdot, \cdot \rangle_{U^1} \) is symmetric. Thus, there exists a unique framed VOA structure on \( U = U^0 \oplus U^1 \) by Proposition 3.6.

By the lemma above, we can apply the extension property of simple current extensions in Theorem 3.7 to define a framed VOA structure on \( V_{4A} \) with structure codes \((\mathcal{C}^0 \cup (\mathcal{C}^1 + \kappa), \mathcal{D}^0 \cup (\mathcal{D}^1 + \xi))\). We know that \( V_{4A} \) is holomorphic by Corollary 3.36. We shall prove
that \( V_{4A} \) is isomorphic to the moonshine VOA \( V_z \). On \( V_{2B} = V^z(1, 1) \oplus V^z(1, -1) \oplus V^z(\theta_\xi^2, 1) \oplus V^z(\theta_\xi^3, -1) \), define \( \psi_1, \psi_2 \in \text{Aut}(V_{2B}) \) by

\[
\psi_1 := \begin{cases} 
1 & \text{on } V^z(1, 1) \oplus V^z(1, -1), \\
-1 & \text{on } V^z(\theta_\xi^2, 1) \oplus V^z(\theta_\xi^3, -1),
\end{cases}
\psi_2 := \begin{cases} 
1 & \text{on } V^z(1, 1) \oplus V^z(\theta_\xi^2, 1), \\
-1 & \text{on } V^z(1, -1) \oplus V^z(\theta_\xi^3, -1).
\end{cases}
\]

Then both of \( \psi_1, \psi_2 \) are involutions on \( V_{2B} \) by the fusion rules in Proposition 3.66.

**Lemma 3.70.** The fixed point subalgebras \( V_{2B}^{(\psi_1)} \) and \( V_{2B}^{(\psi_2)} \) are isomorphic to the \( \mathbb{Z}_2 \)-orbifold subalgebra \( V_\Lambda^+ \) of the Leech lattice VOA.

**Proof:** We have shown that \( V_{2B} \) is isomorphic to the Leech lattice VOA \( V_\Lambda \). By (3.32), the fixed point subalgebra \( V_{2B}^{(\psi_i)} \) is isomorphic to the \( \mathbb{Z}_2 \)-orbifold \( V_\Lambda^+ \). So it remains to prove that \( V_{2B}^{(\psi_2)} \) is isomorphic to \( V_{2B}^{(\psi_1)} \). Since the weight one subspace \( (V_{2B})_1 \) of \( V_{2B} \) is a subspace of \( V^z(\theta_\xi^2, -1) \) by Lemma 3.67, both \( \psi_1 \) and \( \psi_2 \) acts as \(-1\) on \( (V_{2B})_1 \). The weight one subspace of \( V_{2B} \) generates a sub VOA isomorphic to the free bosonic VOA \( M_{CL}(0) \) associated to the linear space \( \mathbb{C} \Lambda = \mathbb{C} \otimes \Lambda \). Since \( \psi_1 \psi_2^{-1} \) trivially acts on the weight one subspace of \( V_{2B} \), \( \psi_1 \psi_2^{-1} \) commute with the action of \( M_{CL}(0) \) on \( V_{2B} \). Therefore, \( \psi_1 \psi_2^{-1} \) is a linear character \( \rho_h = \exp(2\pi \sqrt{-1} h(0)) \in \text{Aut}(V_{2B}) \) induced by a weight one vector \( h \in (V_{2B})_1 \). Since \( \psi_1 = \psi_2 = -1 \) on the weight one subspace, we have \( \psi_1 \rho_h = \rho_{-h} \psi_1 = \rho_h \psi_1 \) for \( i = 0, 1 \). Then \( \psi_1 = \rho_h \psi_2 = \rho_{h/2} \rho_{h/2} \psi_2 = \rho_{h/2} \psi_2 \rho_{h/2}^{-1} \) so that \( \psi_1 \) and \( \psi_2 \) are conjugate in \( \text{Aut}(V_{2B}) \). From this we have the desired isomorphism \( \rho_{h/2} : (V_{2B})^{(\psi_2)} \simeq (V_{2B})^{(\psi_1)} \).

**Corollary 3.71.** There exists \( \rho \in \text{Aut}((V^z)^{(\theta_\xi)}) \) such that \( V^z(1, -1)^\rho \simeq V^z(\theta_\xi^3, 1) \).

**Proof:** Since \( V^z(1, 1) \subset \rho_{h/2} (V_{2B})^{(\psi_i)} \cap (V_{2B})^{(\psi_2)} = (V_{2B})^{(\psi_1)} \cap (V_{2B})^{(\psi_2)} \), \( \rho_{h/2} \) keeps \( (V^z)^{(\theta_\xi)} = V^z(1, 1) \) invariant. Thus the restriction of \( \rho_{h/2} \) on \( (V^z)^{(\theta_\xi)} \) is the desired automorphism.

By the classification of irreducible modules over \( V_\Lambda^+ \) and the fusion rules in Proposition 3.66, the irreducible untwisted \( (V_{2B})^{(\psi_2)} \)-modules are as follows.

\[
V_\Lambda^+ \simeq V^z(1, 1) \oplus V^z(\theta_\xi^2, 1), \quad V_\Lambda^{T+} \simeq V^z(\theta_\xi, 1) \oplus V^z(\theta_\xi^3, 1), \\
V_\Lambda^- \simeq V^z(1, -1) \oplus V^z(\theta_\xi^2, -1), \quad V_\Lambda^{T-} \simeq V^z(\theta_\xi, -1) \oplus V^z(\theta_\xi^3, -1).
\]

Actually, the isomorphisms above are induced by \( \rho \in \text{Aut}((V^z)^{(\theta_\xi)}) \) defined in Corollary 3.71. By the isomorphisms above, the space

\[
V_{4A} = V^z(1, 1) \oplus V^z(\theta_\xi, 1) \oplus V^z(\theta_\xi^2, 1) \oplus V^z(\theta_\xi^3, 1)
\]

is a \( \mathbb{Z}_4 \)-graded simple current extension of \((V^z)^{(\theta_\xi)}\) and isomorphic to \( V^z = V_\Lambda^+ \oplus V_\Lambda^{T+} \) as \((V^z)^{(\theta_\xi)}\)-modules. Since both \( V_{4A} \) and \( V^z \) are simple current extensions of \((V^z)^{(\theta_\xi)}\), these two VOA structures are isomorphic. Therefore, we have obtained our main result in this section.

**Theorem 3.72.** The VOA \( V_{4A} \) obtained by the \( 4A \)-twisted orbifold construction \( V_{4A} \) of \( V^z \) is isomorphic to \( V^z \).
References


