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LOCAL THETA CORRESPONDENCE OF SUPERCUSPIDAL REPRESENTATIONS

\[ F := \text{ a } p\text{-adic field with odd residue characteristic; } \]
\[ f := \text{ the residue field of } F; \]
\[ D := F \text{ or a quadratic extension of } F; \]
\[ \mathcal{O} := \text{ the ring of integers of } D; \]
\[ \varpi := \text{ a prime element of } \mathcal{O}; \]
\[ d := \text{ the residue field of } D \text{ (i.e., } \mathcal{O}/\varpi \mathcal{O}); \]
\[ \tau := \begin{cases} 
\text{the identity map of } F & \text{if } D = F; \\
\text{the involution of } D \text{ over } F & \text{if } D \neq F; 
\end{cases} \]
\[ \epsilon := 1 \text{ or } -1; \]

Let \((V, \langle \cdot, \cdot \rangle)\) be a (non-degenerate) \(\epsilon\)-Hermitian space over \(D\), i.e., the form \(\langle \cdot, \cdot \rangle : V \times V \to D\) satisfies the following conditions:

\[
\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle; \\
\langle xa, yb \rangle = \tau(a)\langle x, y \rangle b; \\
\langle x, y \rangle = \epsilon \tau(\langle y, x \rangle)
\]

for any \(x, y \in V\) and \(a, b \in D\). Let \(U(V)\) be the group of isometries of \((V, \langle \cdot, \cdot \rangle)\).

Then \(U(V)\) is

\[
\begin{cases} 
\text{an orthogonal group } O(V) & \text{if } D = F \text{ and } \epsilon = 1; \\
\text{a symplectic group } Sp(V) & \text{if } D = F \text{ and } \epsilon = -1; \\
\text{a unitary group } U(V) & \text{if } D \text{ is a quadratic extension of } F.
\end{cases}
\]

A representation of the group \(U(V)\) is a pair \((\pi, E)\) where \(E\) is a topological vector space over the complex number field \(\mathbb{C}\) and \(\pi\) is a group homomorphism from \(U(V)\) to the group of all invertible linear transformations on \(E\). A representation satisfying certain smooth condition is called admissible. Parabolic induction is the basic tool to construction admissible representations of reductive groups. However, we can not expect all irreducible admissible representations can be obtained by this way. An irreducible admissible representation \(\pi\) which is not contained in any representations induced from a proper parabolic subgroup is called a supercuspidal representation.
Let \((V, \langle \cdot, \cdot \rangle)\) be an \(\epsilon\)-Hermitian space and \((V', \langle \cdot, \cdot \rangle')\) be an \(\epsilon'\)-Hermitian space such that \(\epsilon \epsilon' = -1\). Let \(W := V \otimes_D V'\) and \(\langle \cdot, \cdot \rangle\) be a form on \(W\) defined by
\[
\langle \cdot, \cdot \rangle := \text{Tr}(\langle \cdot, \cdot \rangle \otimes \tau(\langle \cdot, \cdot \rangle'))
\]
where “\(\text{Tr}\)” denotes the trace from \(D\) to \(F\). Then \(W\) becomes a symplectic space over \(F\) and both \(U(V)\) and \(U(V')\) are subgroups of the symplectic group \(\text{Sp}(W)\). The pair \((U(V), U(V'))\) is called a (type I) reductive dual pair in \(\text{Sp}(W)\). It is known that \(U(V)\) is the centralizer of \(U(V')\) in \(\text{Sp}(W)\) and \(U(V')\) is also the centralizer of \(U(V)\) in \(\text{Sp}(W)\).

The metaplectic group \(\tilde{\text{Sp}}(W)\) has a very special and important representation \(\omega_\psi\) depending on the symplectic form \(\langle \cdot, \cdot \rangle\) on \(W\) and a nontrivial (additive) character \(\psi\) of \(F\). This representation \(\omega_\psi\) is called the Weil representation or the oscillator representation of the metaplectic group \(\tilde{\text{Sp}}(W)\). Let \(\tilde{U}(V)\) be the inverse image of \(U(V)\) in \(\tilde{\text{Sp}}(W)\). For the cases that we consider there exists a splitting map \(U(V) \rightarrow \tilde{U}(V)\) of the extension:
\[
1 \rightarrow \mathbb{C}^\times \rightarrow \tilde{U}(V) \rightarrow U(V) \rightarrow 1
\]
Hence, we can regard \(U(V)\) as a subgroup of \(\tilde{\text{Sp}}(W)\) via the splitting.

The map
\[
\rho: U(V) \times U(V') \rightarrow U(V)U(V')
\]
by \((g, g') \mapsto gg'\) is group homomorphism. Restrict the Weil representation \(\omega_\psi\) to \(U(V)U(V')\) and pull back to \(U(V) \times U(V')\) via \(\rho\). Then we can regard \(\omega_\psi\) as a representation of \(U(V) \times U(V')\). An irreducible admissible representation \(\pi\) of \(U(V)\) is said to correspond to an irreducible admissible representation \(\pi'\) of \(U(V')\) if there is a nontrivial \(U(V) \times U(V')\)-map
\[
\omega_\psi \mapsto \pi \otimes \pi'.
\]
This establishes a correspondence between some irreducible admissible representations of \(U(V)\) and some irreducible admissible representations of \(U(V')\). This correspondence is called local theta correspondence or Howe duality.

**Theorem (Howe-Waldspurger).** The local theta correspondence is one-to-one (when \(p\) is odd).
The fundamental problem is to investigate the explicit theta correspondence. That is, for a given irreducible admissible representation \( \pi \) of \( U(V) \) we hope to understand the following two questions:

1. In what condition of \( V' \) does \( \pi \) occur in the theta correspondence? (occurrence)
2. Which representation \( \pi' \) of \( U(V') \) is \( \pi \) paired with once \( \pi \) occurs? (correspondence)

A two-dimensional \( \epsilon \)-Hermitian space is called a hyperbolic plane if it admits one-dimensional subspace of isotropic vectors (i.e., \( \langle v, v \rangle = 0, v \neq 0 \)). Suppose that \( V_0 \) is an anisotropic \( \epsilon \)-Hermitian space (i.e., having no isotropic vectors) and \( V_k \) is the direct sum of \( V_0 \) and \( k \) copies of hyperbolic planes. The number \( k \) is called the Witt index of \( V_k \). The space \( V_k \) is embedded into \( V_{k+1} \) by \( v \mapsto (v, 0, 0) \) for \( v \in V_k \). Then the series

\[
V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots
\]

is called a Witt tower.

The following properties are well-known and are called the induction principle of the theta correspondence:

1. Let \( \pi \) be an irreducible admissible representation of \( U(V) \). Suppose that \( \pi \) occurs in the theta correspondence for the dual pair \( (U(V), U(V_{n'}')) \). Then the representation \( \pi \) occurs in the theta correspondence for the dual pair \( (U(V), U(V_{k}'')) \) for any \( k' \geq n' \).
2. If the dimension of \( V \) is less than or equal to the Witt index of \( V' \), then every irreducible admissible representation of \( U(V) \) occurs in the theta correspondence for \( (U(V), U(V')) \).
3. (by S. Kudla) Let \( \pi \) be an irreducible supercuspidal representation of \( U(V) \). Suppose that \( \pi \) occurs in the theta correspondence for the reductive dual pair \( (U(V), U(V_{n'}')) \) and corresponds to \( \pi' \) but never occurs in the theta correspondence for the pair \( (U(V), U(V_{k}'')) \) for any \( k' < n' \). Then \( \pi' \) is also a supercuspidal representation.

From 3, we can talk about the first occurrence of supercuspidal representations. Suppose \( V \) is fixed. We consider two related Witt towers \( V'^{\uparrow+}_0 \subset V'^{\uparrow+}_1 \subset V'^{\uparrow+}_2 \subset \cdots \) and \( V'^{\downarrow-}_0 \subset V'^{\downarrow-}_1 \subset V'^{\downarrow-}_2 \subset \cdots \) as follows:

1. \( V \) is symplectic and \( V'^{\uparrow+}_0 \) (resp. \( V'^{\downarrow-}_0 \)) is a trivial quadratic space (resp. a four-dimensional anisotropic quadratic space).
(ii) $V$ is symplectic, both $V'_0^+$ and $V'_0^-$ are two-dimensional anisotropic spaces with non-isomorphic quadratic forms.

(iii) $V$ is an even-dimensional quadratic space, both $V'_0^+$ and $V'_0^-$ are the trivial symplectic space.

(iv) $V$ is unitary, and $V'_0^+$ (resp. $V'_0^-$) is a trivial unitary space (resp. a two-dimensional anisotropic unitary space).

(v) $V$ is unitary, both $V'_0^+$ and $V'_0^-$ are one-dimensional anisotropic spaces with non-isomorphic $\epsilon'$-Hermitian forms.

Define the “sgn” character of $U(V)$ as follows. If $U(V)$ is the trivial group or a symplectic group, let sgn be the trivial character of $U(V)$. If $U(V)$ is a (nontrivial) orthogonal group (resp. unitary group), let sgn be the character of order two whose restriction to the special orthogonal group (resp. special unitary group) is trivial.

**Conjecture (Kudla-Rallis).** Suppose that $\pi$ is an irreducible admissible representation of $U(V)$. Let $\ell^+(\pi)$ (resp. $\ell^-(\pi \otimes \text{sgn})$) denote the smallest dimension of $V_{k^+}^+$ (resp. $V_{k^-}^-$) such that $\pi$ (resp. $\pi \otimes \text{sgn}$) occurs in the theta correspondence for the pair $(U(V), U(V_{k^+}^+))$ (resp. $(U(V), U(V_{k^-}^-))$). Then

$$
\ell^+(\pi) + \ell^-(\pi \otimes \text{sgn}) = \begin{cases} 
2 \dim(V) + 4, & \text{for cases (i), (ii);} \\
2 \dim(V), & \text{for case (iii);} \\
2 \dim(V) + 2, & \text{for cases (iv), (v).}
\end{cases}
$$

Each of $\ell^+(\pi)$ and $\ell^-(\pi \otimes \text{sgn})$ depends on $\pi$ but their sum does not. This conjecture is the so called *preservation principle* of the theta correspondence. This conjecture is proved to be true for the following situations (among some other special cases):

1. $U(V)$ is a unitary group (i.e., cases (iv) and (v)), the dimensions of $V$ and $V_{k^+}^+$, $V_{k^-}^-$ are of the same parity (i.e., all even or all odd) and $\pi$ is supercuspidal (by S. Kudla and S. Rallis)

2. $\pi$ is a supercuspidal representation of depth zero (by P.).

For each $V'^+$ or $V'^-$, there are still two related towers $V^+$ and $V^-$. The conjecture of preservation principle suggests that there exist a sequence of dimensions $n = n_0 < n_1 < n_2 < \cdots < n_i < \cdots$ of $V^\pm$ or $V'^\pm$ and irreducible supercuspidal representations $\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots$ of $U(V^\pm)$ or $U(V'^\pm)$ such that $\pi_i \otimes \text{sgn}$ and $\pi_{i+1}$ are paired.
by the theta correspondence. Suppose that $\pi_0$ does not come from a smaller group via the theta correspondence.

1. If $U(V)$ is orthogonal, then the sequence of the dimensions is
   \[ n, n, n + 4, n + 8, n + 16, \ldots, n + i(i + 1) + (-1)^i \left\lfloor \frac{i}{2} \right\rfloor, \ldots \]

2. If $U(V)$ is symplectic, then the sequence of the dimensions is
   \[ n, n + 4, n + 8, n + 16, n + 24, \ldots, n + i(i + 1)(i + 2) + (-1)^{i+1} \left\lfloor \frac{i+1}{2} \right\rfloor, \ldots \]

3. If $U(V)$ is unitary and the dimensions of $V^\pm$ and $V'^\pm$ are of the same parity, then the sequence of the dimensions is
   \[ n, n + 2, n + 6, n + 12, n + 20, \ldots, n + i(i + 1), \ldots \]

4. If $U(V)$ is unitary and the dimensions of $V^\pm$ and $V'^\pm$ are of the opposite parity, then the sequence of the dimensions is
   \[ n, n + 1, n + 4, n + 9, n + 16, \ldots, n + i^2, \ldots \]

**Problem:** Suppose $\pi_0$ is given. How to construct $\pi_i$?

Let $v$ be a (non-degenerate) $\epsilon$-Hermitian space over $d$ and $U(v)$ be the group of isometries. Then $U(v)$ is a finite classical group (orthogonal, symplectic or unitary). P. Deligne and G. Lusztig defined a very special class of representations of $U(v)$ called *unipotent cuspidal*. Not every $U(v)$ has a unipotent cuspidal representation. If $U(v)$ has a unipotent cuspidal representation, then the dimension of $v$ must be of the form:

\[
\dim(v) = \begin{cases} 
2i^2 & \text{for some } i \text{ if } U(v) \text{ is an even orthogonal group}; \\
2i(i + 1) & \text{for some } i \text{ if } U(v) \text{ is a symplectic group}; \\
\frac{i(i+1)}{2} & \text{for some } i \text{ if } U(v) \text{ is a unitary group}; 
\end{cases}
\]

Let $v$ be a fixed $\epsilon$-Hermitian space over $d$. We consider two related Witt towers $v_0^+ \subset v_1^+ \subset v_2^+ \subset \cdots$ and $v_0^- \subset v_1^- \subset v_2^- \subset \cdots$ as follows:

(i) $v$ is symplectic and $v_0^+$ (resp. $v_0^-$) is a trivial quadratic space (resp. a two-dimensional anisotropic quadratic space).

(ii) $v$ is symplectic, both $v_0^+$ and $v_0^-$ are one-dimensional anisotropic spaces with non-isomorphic quadratic forms.

(iii) $v$ is a quadratic space, both $v_0^+$ and $v_0^-$ are the trivial symplectic space.
(iv) $v$ is unitary, and $v_{0}^{+}$ (resp. $v_{0}^{-}$) is a trivial unitary space (resp. a one-dimensional anisotropic unitary space).

**Conjecture.** Suppose that $\eta$ is an irreducible representation of $U(v)$. Let $\ell^{+} (\eta)$ (resp. $\ell^{-} (\eta \otimes \text{sgn})$) denote the smallest dimension of $v_{k}^{+}$ (resp. $v_{k}^{-}$) such that $\eta$ (resp. $\eta \otimes \text{sgn}$) occurs in the theta correspondence for the finite dual pair $(U(v), U(v_{k}^{+}))$ (resp. $(U(v), U(v_{k}^{-}))$). Then

$$\ell^{+} (\eta) + \ell^{-} (\eta \otimes \text{sgn}) = \begin{cases} 2 \dim(v) + 2, & \text{for cases (i), (ii)}; \\ 2 \dim(v), & \text{for case (iii)}; \\ 2 \dim(v) + 1, & \text{for case (iv)}. \end{cases}$$

The conjecture is proved (by P.) when $\eta$ is a cuspidal representation.

The preservation principle for finite reductive dual pairs suggests that there exist a sequence of dimensions $m = m_{0} < m_{1} < m_{2} < \cdots < m_{i} < \cdots$ of $v^{\pm}$ or $v'^{\pm}$ and irreducible cuspidal representations $\eta = \eta_{0}, \eta_{1}, \eta_{2}, \ldots, \eta_{i}, \ldots$ of $U(v^{\pm})$ or $U(v'^{\pm})$ such that $\eta_{i} \otimes \text{sgn}$ and $\eta_{i+1}$ are paired by the theta correspondence. Suppose that $\eta_{0}$ does not come from a smaller group via the theta correspondence.

1. If $U(v)$ is orthogonal and $m$ is even, then the sequence of the dimensions is

$$m, m, m + 2, m + 4, m + 8, m + 12, \ldots, m + \frac{i(i + 1)}{2} - \left\lfloor \frac{i}{2} \right\rfloor, \ldots$$

2. If $U(v)$ is orthogonal and $m$ is odd, then the sequence of the dimensions is

$$m, m + 1, m + 4, m + 7, m + 12, \ldots, m + \frac{i(i + 1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor, \ldots$$

3. If $U(v)$ is unitary, then the sequence of the dimensions is

$$m, m + 1, m + 3, m + 6, m + 10, \ldots, m + \frac{i(i + 1)}{2}, \ldots$$

When $m = 0$ in cases 1 or 3 (i.e., $\eta_{0}$ is the trivial representation of the trivial group), then the chain of cuspidal representations are exactly the chain of unipotent cuspidal representations. When $m = 1$ and $\eta_{0}$ is the trivial representation in case 2, the even terms in the sequence are exactly the unipotent cuspidal representations of the finite symplectic groups.

From now we assume that $D$ is an unramified quadratic extension of $F$. So $U(V)$ is a unitary group. Let $L$ be a lattice in $V$ i.e., a (free) $\mathcal{O}$-module whose rank is equal to the dimension of $V$. Define

$$L^{*} = \{ v \in V \mid \langle v, l \rangle \in \mathcal{O} \text{ for all } l \in L \}.$$
$L^*$ is also a lattice in $V$. The lattice $L$ is called a good lattice if

$$L^* \subseteq L \subseteq L^*.$$  

Let $L$ be a good lattice. Then $v^* := L^*/L$ and $v := L/L^*$ are vector spaces over $d$ with non-degenerate forms. Define

$$U(V)_L := \{ g \in U(V) \mid g.L = L \}.$$  

Then $U(V)_L$ is a maximal open compact subgroup of $U(V)$ and there is a surjective homomorphism:

$$U(V)_L \to U(v) \oplus U(v^*).$$

Note that $\dim(V) = \dim(v) + \dim(v^*)$. If $\eta$ and $\eta^*$ are representations of $U(v)$ and $U(v^*)$ respectively, then $\eta \otimes \eta^*$ can be pulled back as a representation of $U(V)_L$ via the above homomorphism.

Suppose $D$ is an unramified quadratic extension of $F$. Suppose $\pi_0$ is a supercuspidal representation of $U(V_{k_0})$ which does not come from a smaller group via theta correspondence. By (the conjecture of) the preservation principle there exists a chain of supercuspidal representation $\pi_0, \pi_1, \pi_2, \ldots$ of $U(V_{k_0}), U(V_{k_1}), U(V_{k_2}), \ldots$ such that $\pi_i \otimes \text{sgn}$ and $\pi_{i+1}$ are paired in the theta correspondence for each $i$. It is known that either $\dim(V_{k_i}) = \dim(V_{k_0}) + i(i + 1)$ (called case 1) or $\dim(V_{k_i}) = \dim(V_{k_0}) + i^2$ (called case 2).

**Theorem (P).** Suppose $\pi_0$ is of depth 0. It is known that $\pi_0$ is of the form

$$\pi_0 = \text{Ind}_{U(V_{k_0})}^{U(V_{k_0})} (\eta_0 \otimes \eta'_0)$$

for some good lattice $L_0$ in $V_{k_0}$ and cuspidal representations $\eta_0, \eta'_0$. Then the representation $\pi_i$ is isomorphic to the induced representation

$$\text{Ind}_{U(V_{k_i})}^{U(V_{k_i})} (\eta_i \otimes \eta'_i),$$

where $L_i$ is a good lattice in $V_{k_i}$ and $\eta_i, \eta'_i$ are the $i$-th term of cuspidal representation in the chain starting from $\eta_0, \eta'_0$ respectively.

J.-K. Yu defines some data to construction irreducible supercuspidal representations (for the “tame” case) as follows. Let $F^t$ denote the maximal tamely ramified extension of $F$.  

Let \((\vec{G}, \xi_0, \vec{\phi})\) be a triple where \(\vec{G} := (G^0, \ldots, G^d)\) is a tower of algebraic subgroups of a connected reductive group \(G\) over \(F\),

\[
G^0 \subset G^1 \subset \cdots \subset G^d = G,
\]
such that \(Z(G^0)/Z(G)\) is anisotropic and each \(G^j \otimes F^t\) is split and is a Levi factor of a parabolic subgroup of \(G \otimes F^t\), \(\xi_0\) is a supercuspidal representation of \(G^0(F)\) of depth zero, and \(\vec{\phi} := (\phi_0, \ldots, \phi_d)\) is such that \(\phi_j\) is a linear character of \(G^j(F)\) for each \(j\) such that either

(i) \(0 < \text{depth}(\phi_0) < \cdots < \text{depth}(\phi_d)\); or

(ii) \(0 < \text{depth}(\phi_0) < \cdots < \text{depth}(\phi_{d-1})\) and \(\phi_d\) is trivial.

Then J.-K. Yu constructs an open subgroup \(K_d\) which is compact modulo the center of \(G(F)\) and an irreducible representation \(\rho_d\) of \(K_d\).

**Theorem (J.-K. Yu).** Suppose that and \(\phi_j\) is \(G^j+1\)-generic for \(0 \leq j \leq d - 1\). Then the induced representation \(\pi = \text{Ind}_{K_d}^{G(F)} \rho_d\) is an irreducible supercuspidal representation of \(G(F)\).

A \((\vec{G}, \pi_0, \vec{\phi})\) datum satisfying above theorem is called a *generic datum*. The supercuspidal representation constructed above is of depth equal to \(\text{depth}(\phi_d)\) (or \(\text{depth}(\phi_{d-1})\)).

It is expected (but is still unknown) that all irreducible supercuspidal representations of \(G(F)\) if the residue characteristic of \(F\) is large enough.

**“Theorem” (P.).** Suppose \(\pi\) is the irreducible supercuspidal representation of \(U(V) = U(V_{k_0})\) where \(V\) is odd-dimensional constructed from the (generic) datum \((\vec{G}, \xi_0, \vec{\phi})\) such that \(0 < \text{depth}(\phi_0) < \cdots < \text{depth}(\phi_d)\).

(i) Then \(\pi\) does not come from a smaller group (i.e., \(\dim(V) \leq \dim(V')\) when \(\pi\) occurs).

(ii) The \(i\)-th term \(\pi_i\) of the chain of supercuspidal representations starting from \(\pi = \pi_0\) is constructed from the datum \((\vec{G}^i, \xi_0, \vec{\phi}^i)\) where \(\vec{G}^i = (G^{0}, \ldots, G^{d+i+1})\) and \(\vec{\phi}^i = (\phi'_0, \ldots, \phi'_{d+1})\) such that

\(1\) \(G^j = G^j \times G^*\) for \(j = 0, \ldots, d\) and \(G^{d+i+1} = G'\) where \(G^*\) (resp. \(G'\)) is the algebraic group such that \(G^*(F) = U(V_{k_0}^2)\) (resp. \(G'(F) = U(V_{k_0})\)) and \(V_{k_0} = V_{k_0}^1 \oplus V_{k_0}^2\) such that \(U(V_{k_0}^1) \simeq U(V_{k_0})\) and \(\dim(V_{k_0}^2) = i(i + 1)\) (case 1) or \(i^2\) (case 2);
(2) $\phi_j'$ is the linear character $\phi_j'' \otimes \text{triv}$ of $G^j(F) \times G^*(F)$ for $j = 0, \ldots, d$ and $\phi_{d+1}'$ is trivial where $\phi_j''$ is the contragredient character $\tilde{\phi}_j$ if $i$ is odd, and is $\phi_j$ if $i$ is even;

(3) $\xi_0' = \xi_0 \otimes \xi_0''$ where $\xi_0''$ is the depth zero supercuspidal representation induced from $\eta_i \otimes \eta_i$ (case 1) or $\eta_i \otimes \eta_{i-1}$ (case 2) and $\eta_i$ is the (unique) unipotent cuspidal representation of finite unitary group in $\frac{(i+1)}{2}$ variables.

Example.

(i) Let $V_0$ (resp. $V_1$) be a one-dimensional $\epsilon$-Hermitian space (resp. two-dimensional $\epsilon'$-Hermitian with Witt index one) over a quadratic extension $D$ of $F$. Suppose that a character $\pi_0$ of $U(V_0)$ of positive depth is paired with $\pi_1$ of $U(V_1)$. We know $\pi_1$ is supercuspidal. We can show that there is a decomposition $V_1 = V_1^1 \oplus V_1^2$ such that the restriction of $\pi_1$ to $H_1 := U(V_1^1) \times U(V_1^2)$ contains $\tilde{\pi}_0 \otimes \text{triv}$ where $\tilde{\pi}_0$ is the contragredient representation of $\pi_0$. J. Adler define a subgroup $J_1$ normalized by $H_1$ and he constructs an irreducible representation $\rho_{\tilde{\pi}_0}$ of $H_1J_1$ whose restriction to $H_1$ contains $\tilde{\pi}_0 \otimes \text{triv}$ and $\text{Ind}_{H_1J_1}^{U(V_1)} \rho_{\tilde{\pi}_0}$ is irreducible supercuspidal. Then we know that the restriction of $\pi_1$ to $H_1J_1$ contains $\rho_{\tilde{\pi}_0}$. Hence, we conclude that $\pi_1$ is isomorphic to $\text{Ind}_{H_1J_1}^{U(V_1)} \rho_{\tilde{\pi}_0}$.

(ii) Let $V_2$ be a 5-dimensional $\epsilon$-Hermitian space with Witt index two. Suppose that the irreducible representation $\pi_2$ of $U(V_2)$ is paired with $\pi_1 \otimes \text{sgn}$ of $U(V_1)$. We can show that there is a decomposition $V_2 = V_2^1 \oplus V_2^2$ such that $V_2^1$ is one-dimensional and the restriction of $\pi_2$ to $H_2 := U(V_2^1) \times U(V_2^2)_{L_2}$ contains the representation $(\pi_0 \otimes \text{sgn}) \otimes (\eta_1 \otimes \eta_2)$ where $\eta_1$ (resp. $\eta_2$) is the unipotent cuspidal representation of finite unitary group in one (resp. three) variable(s) and $\eta_1 \otimes \eta_2$ is regarded as a representation of $U(V_2^2)_{L_2}$ via the map $U(V_2^2)_{L_2} \rightarrow U(L_2/L_2^\omega) \times U(L_2^\omega/L_2)$. We can construct an irreducible admissible representation $\rho_2$ of $H_2J_2$ whose restriction to $H_2$ contains $(\pi_0 \otimes \text{sgn}) \otimes (\eta_1 \otimes \eta_2)$. Then we conclude that $\pi_2$ is isomorphic to $\text{Ind}_{H_2J_2}^{U(V_2)} \rho_2$. 