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ASYMPTOTIC BEHAVIOR OF THE
COMPRESSIBLE VISCOUS POTENTIAL FLUID:
RENORMALIZATION GROUP APPROACH

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Abstract. In this article we apply the renormalization group method to
study the potential flows of a compressible viscous fluid at small Reynolds
number. The derived renormalization equation of order one is a system of
reaction convection diffusion equations. The global existence and unique-
ness of the weak solutions satisfying the energy inequality are proved
following the methodology of Leray. The comparison between the exact
solution and its approximation is also discussed.

Keywords: Slight compressible potential flow, renormalization group, wave
group, reaction convection diffusion system, incompressible limit.

\section{Introduction}

In this paper we apply the renormalization group method to study the
asymptotic behaviour of solutions of the potential flows of a compressible vis-
cous fluid at small Reynolds number when the Mach number tends to zero
in a suitable scaling. More precisely, we consider for all $\varepsilon > 0$ the system of
equations

\begin{align}
\partial_t \varphi_\varepsilon + \frac{1}{\varepsilon^2} (\rho_\varepsilon - 1) &= \Delta \varphi_\varepsilon, \\
\partial_t \rho_\varepsilon + \text{div} (\rho_\varepsilon \nabla \varphi_\varepsilon) &= 0, \quad \text{in } Q_T = \Omega \times (0, T).
\end{align}
where $\Omega$ is an open set in $\mathbb{R}^N$ and $0 < T < \infty$. Here $\rho_\epsilon > 0$ is the fluid density and $u_\epsilon = \nabla \varphi_\epsilon$ is fluid velocity; $\nabla$ and $\text{div}$, the gradient and divergence operators with respect to the space variables; $\Delta$, the Laplace operator. The factor 1 is a consequence of the normalization as we will see in the assumption (1.3). Note that $\epsilon \in (0, 1)$ is a small parameter related to the Mach number.

The unknown functions are assumed to take prescribed values at initial time,

$$
\rho_\epsilon(x, 0) = \rho_\epsilon^0(x) \geq 0, \quad \varphi_\epsilon(x, 0) = \varphi_\epsilon^0(x), \quad x \in \Omega, \quad (1.2)
$$

$$
\int_\Omega \rho_\epsilon^0(x) dx = 1, \quad \int_\Omega \varphi_\epsilon^0(x) dx = 0. \quad (1.3)
$$

We also impose the following natural requirements on the solution

$$
\rho_\epsilon(x, t) \geq 0, \quad (x, t) \in Q_T, \quad (1.4)
$$

$$
\int_\Omega \rho_\epsilon(x, t) dx = \int_\Omega \rho_\epsilon^0(x) dx, \quad (x, t) \in Q_T, \quad t \in [0, T].
$$

The existence of global weak solution to (1.1) – (1.4) for a fixed Mach number $\epsilon > 0$, say $\epsilon = 1$ for example, obtained by Vaigant and Kazhikhov [28] (see also [21] for further discussion). They also showed that the weak solution is smooth in the two-dimensional case provided that the initial and boundary data are smooth. The incompressible limit of the potential flows of a compressible viscous fluid at small Reynolds number is studied in [19]. It is shown that the singular limit system for the potential flow of the viscous compressible fluid dynamics equations as the Mach number tends to zero is the Laplace equation combined with the linear part of the Bernoulli equation.

Using the wave group methods introduced by Schochet [25] and Grenier [11] we obtain the strong compactness of the first corrector of the density, i.e., the acoustic wave which is also shown to satisfy the transport equation which describes the convection by incompressible velocity field of a scalar quantity (see [19]).

As was shown in [19], the compressible potential flow equations (1.1a, b) rewrites in terms of the density fluctuation $g_\epsilon = (\rho_\epsilon - 1)/\epsilon, \varphi_\epsilon$ and $w_\epsilon = (\varphi_\epsilon, g_\epsilon)^t$ as follows

$$
\frac{d}{dt} w_\epsilon + \frac{1}{\epsilon} L w_\epsilon = \mathcal{F}(w_\epsilon), \quad w_\epsilon(0) = w_\epsilon^{\text{in}} \equiv (\varphi_\epsilon^{\text{in}}, g_\epsilon^{\text{in}})^t. \quad (1.5)
$$

Here $L$ is the typical linear wave operator

$$
L \begin{pmatrix} \varphi \\ g \end{pmatrix} \equiv \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \varphi \\ g \end{pmatrix} \quad (1.6)
$$

and $\mathcal{F}$ is a nonlinear operator defined by

$$
\mathcal{F}(w_\epsilon) \equiv \begin{pmatrix} \Delta \varphi_\epsilon \\ -\text{div}(g_\epsilon, \nabla \varphi_\epsilon) \end{pmatrix} \quad (1.7)
$$
Yet, another formulation for (1.1a,b) is given by introduce the velocity \( u_\epsilon = \nabla \varphi_\epsilon \). Denote \( \tilde{\omega}_\epsilon = (u_\epsilon, g_\epsilon)^t \) then we have

\[
\frac{d}{dt} \tilde{\omega}_\epsilon + \frac{1}{\epsilon} \tilde{L} \tilde{\omega}_\epsilon = \tilde{\mathbf{F}}(\tilde{\omega}_\epsilon), \quad \tilde{\omega}_\epsilon(0) = \tilde{\omega}^{in} \equiv (u_{\epsilon}^{in}, g_{\epsilon}^{in})^t.
\]

(1.8)

where \( \tilde{L} \) and \( \tilde{\mathbf{F}} \) are now given respectively by

\[
\tilde{L} \begin{pmatrix} u \\ g \end{pmatrix} \equiv \begin{pmatrix} \nabla g \\ \text{div} u \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ \text{div} & 0 \end{pmatrix} \begin{pmatrix} u \\ g \end{pmatrix}
\]

(1.9)

\[
\tilde{\mathbf{F}}(\tilde{\omega}_\epsilon) \equiv \begin{pmatrix} \Delta u_\epsilon \\ -\text{div}(g_\epsilon u_\epsilon) \end{pmatrix}
\]

(1.10)

The renormalization group approach we use here follows the ideal of Moise, Temam and Ziane [23, 24]. In particular, they simplify the renormalization group method introduced by Chen, Goldenfeld and Oono in [1,2] and apply it to slightly compressible fluid, Swift-Hohenberg and incompressible Navier-Stokes equations. One remarkable observation is that the renormalized equation they obtained is similar to that obtained by applying a Poincaré-Dulac type theory to the equations. For a discussion on the physical motivations and on the mathematical setting up the reader is referred to [23, 24, 30], and references there in.

The perturbative renormalization group has been developed by Chen, Goldenfeld and Oono ([1,2]) as a unified tool for global asymptotic analysis. They solved the problems of continuum mechanics by the traditional method of the renormalization group, as used by physicists. Moreover, using the method of singular expansions around the simple solution where the asymptotics are regular, Goldenfeld and his colleagues were able to obtain instructive and useful approximate solutions for some important problems in turbulence, flow in porous media, and other areas.

The outline of this paper is as follows. In section 2, we briefly review the renormalization group method and derive the renormalization group of (1.5) – (1.7). We also apply the same method to obtain the renormalization group of (1.8) – (1.10) in section 3. Section 4 is devoted to the global existence and uniqueness of the weak solution of the renormalization group equation obtained in section 2 and 3. In the final section, section 5, we estimate the difference between the exact and asymptotic solutions.

§2 Derivation of the renormalization group equation

To begin with, let us briefly introduce the renormalization group method developed by Chen, Goldenfeld and Oono [1,2]. For a mathematically rigorous setting up we referred to Moise, Temam and Ziane [23, 24, 30]. Introducing the time scale \( s = t/\epsilon \) and setting

\[
\psi_\epsilon(s) = \varphi_\epsilon(\epsilon s), \quad q_\epsilon(s) = g_\epsilon(\epsilon s),
\]

(2.1)
we can rewrite (1.5) – (1.7) in the weakly nonlinear problem for \( U_\varepsilon = (\varphi_\varepsilon, q_\varepsilon)^t \)

\[
\frac{d}{ds} U_\varepsilon + L U_\varepsilon = \varepsilon F(U_\varepsilon), \quad U_\varepsilon(0) = U^{in} = (\varphi^{in}, q^{in})^t.
\] (2.2)

The general idea behind the renormalization group approach is to solve the problem iteratively. As noted in [1,2], the advantage of the renormalization group method is that the starting point is a straightforward naive perturbation expansion, for which very little a priori knowledge is required. Proceeding formally, we write the naive asymptotic expansion for \( U = U_\varepsilon \):

\[
U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots.
\] (2.3)

Plugging this ansatz in equation (2.2) and identifying different powers of \( \varepsilon \) yields a cascade of equations

\[
- \frac{d}{ds} U_0 + LU_0 = 0, \quad \text{(2.4a)}
\]

\[
\frac{d}{ds} U_1 + LU_1 = F(U_0), \quad \text{(2.4b)}
\]

\[
\frac{d}{ds} U_2 + LU_2 = \nabla_U F(U_0) \cdot U_1, \quad \text{(2.4c)}
\]

and so on. Introducing the wave group operator \( t \mapsto \mathcal{L}(t) = \exp(-tL) \), we write the solution of (2.4a) with initial condition \( U^{in} \), which can be different from the initial condition of the original problem, in the form

\[
U_0(s) = e^{-sL} U^{in} = \mathcal{L}(s) U^{in}. \quad \text{(2.5)}
\]

Using the Duhamel's formula, we deduce from (2.4b) that

\[
U_1(s) = \mathcal{L}(s) \int_0^s \mathcal{L}(-\tau) F(\mathcal{L}(\tau) U^{in}) d\tau, \quad U_1(0) = 0. \quad \text{(2.6)}
\]

The renormalization group method consists of finding an approximate solution constructed with the help of the solution of a renormalization group equation. Now we apply the renormalization group method to split the solution \( U_1 \). Indeed, we have to decompose the integrand in (2.6) into the resonant and nonresonant parts;

\[
\mathcal{L}(-\tau) F(\mathcal{L}(\tau) U^{in}) = F_R(U^{in}) + F_2(\tau, U^{in}). \quad \text{(2.7)}
\]

where the resonant part \( F_R \) is chosen to satisfy

\[
\mathcal{L}(-\tau) F_R(\mathcal{L}(\tau) U^{in}) = F_R(U^{in})
\]

and the renormalization group equation (2.13) below. Defining

\[
\int_0^s F_2(\tau, U^{in}) d\tau = F_{osc}(s, U^{in}). \quad \text{(2.8)}
\]

then we can represent \( U_1 \) in (2.6) as

\[
U_1(s) = \mathcal{L}(s) \left[ sF_R(U^{in}) + F_{osc}(s, U^{in}) \right]. \quad \text{(2.9)}
\]
Therefore, from (2.5) and (2.9), we can write the naive perturbation expansion (2.3) for \( U = U_\epsilon \) as

\[
U_\epsilon(s) \approx \mathcal{L}(s)[U^{\text{in}} + \epsilon s \mathcal{F}_R(U^{\text{in}}) + \epsilon \mathcal{F}_{\text{osc}}(s, U^{\text{in}})] + O(\epsilon^2).
\] (2.10)

Since \( \epsilon s = t \), the second term in the bracket of (2.10) is of order \( O(1) \). This leads to look for a function \( W \) such that its Taylor expansion of order 1, at \( s = 0 \), coincides with the term \( U^{\text{in}} + \epsilon s \mathcal{F}_R(U^{\text{in}}) \). Thus we have to consider the following problem

\[
\frac{dW}{ds} = \epsilon \mathcal{F}_R(W), \quad W(0) = U^{\text{in}}.
\] (2.11)

Replacing \( U^{\text{in}} \) by \( W \) in (2.10), we have the approximate solution

\[
\mathcal{U}_\epsilon(s) = \mathcal{L}(s)[W(s) + \epsilon \mathcal{F}_{\text{osc}}(s, W(s))].
\] (2.12)

Returning to the slow scale \( t \), (2.11) and (2.12) can be rewritten as

\[
\frac{dW}{dt} = \mathcal{F}_R(W), \quad W(0) = U^{\text{in}},
\] (2.13)

and

\[
\mathcal{U}_\epsilon(t) = \mathcal{L}(t/\epsilon)[W(t) + \epsilon \mathcal{F}_{\text{osc}}(t/\epsilon, W(t))].
\] (2.14)

Eq. (2.13) is called the renormalization group equation (of order 1). The same idea can be applied to obtain the higher order renormalization group equation. For the sake of simplicity, we only renormalized the first order term in the naive perturbation expansion. The formal computation shows that \( \mathcal{U} \) satisfies (1.5) with a perturbed force:

\[
\frac{d\mathcal{U}_\epsilon}{dt} + \frac{1}{\epsilon} L \mathcal{U}_\epsilon = \mathcal{F}(\mathcal{U}_\epsilon) + \epsilon R_\epsilon,
\] (2.15)

where

\[
\epsilon R_\epsilon = \mathcal{F} \left( \mathcal{L}(t/\epsilon) W(t) \right) - \mathcal{F}(\mathcal{U}_\epsilon) + \epsilon \mathcal{L}(t/\epsilon) \left( \nabla_W \mathcal{F}_{\text{osc}} \mathcal{F}_R(W) \right)
\] (2.16)

We now make explicitly the renormalization group equation derived above for (1.5) – (1.7). Let \( U_0 = (\psi_0, q_0)^t \), then (2.4a) reads as

\[
\frac{d}{ds} \psi_0 + q_0 = 0 \quad (2.17a)
\]

\[
\frac{d}{ds} q_0 + \Delta \psi_0 = 0 \quad (2.17b)
\]

which turns out to be the wave equation

\[
\frac{\partial^2}{\partial s^2} q_0 - \Delta q_0 = 0, \quad \frac{\partial^2}{\partial s^2} \psi_0 - \Delta \psi_0 = 0.
\] (2.18)

Introducing the Fourier expansion
\[ q_0 = \sum_k q_{0,k} e^{ikx}, \quad \psi_0 = \sum_k \psi_{0,k} e^{ikx}, \]
\[ \varphi^{in} = \sum_k \varphi_k^{in} e^{ikx}, \quad g^{in} = \sum_k g_k^{in} e^{ikx}, \] 
we have an infinite system of ordinary differential equations
\[ \frac{d\psi_{0,k}}{ds} + q_{0,k} = 0 \] 
\[ \frac{dq_{0,k}}{ds} - |k|^2 \psi_{0,k} = 0 \]
(2.20a)
(2.20b)
For \( k = 0 \) we find
\[ q_{0,0} = g_0^{in} = 0, \quad \text{and} \quad \psi_{0,0} = -sg_0^{in} = 0. \]
(2.21)
Here we use the fact that
\[ \int_\Omega \varphi^{in}(x)dx = \int_\Omega g^{in}(x)dx = 0. \]
If \( k \neq 0 \) we set
\[ R_k = \begin{pmatrix} 0 & 1 \\ -|k|^2 & 0 \end{pmatrix}, \]
(2.22)
then the associated exponential operator is given by
\[ e^{sR_k} = \cos |k|sI + \frac{\sin |k|s}{|k|} R_k = \frac{1}{2} \sum_{\alpha=\pm1} e^{i\alpha|k|s} R_{k,\alpha}, \]
(2.23)
where \( I \) is the \( 2 \times 2 \) identity matrix and
\[ R_{k,\alpha} = \begin{pmatrix} 1 & -i\alpha/|k| \\ i\alpha/|k| & 1 \end{pmatrix}, \quad \alpha = 1, -1. \]
(2.24)
Thus, the solution of (2.20a, b), with initial condition \((\varphi^{in}, g^{in})^t\), is given explicitly by
\[ \begin{pmatrix} \psi_{0,k} \\ q_{0,k} \end{pmatrix} = e^{-sR_k} \begin{pmatrix} \varphi_k^{in} \\ g_k^{in} \end{pmatrix} = \frac{1}{2} \sum_{\alpha=\pm1} e^{i\alpha|k|s} Y_{k,\alpha} \begin{pmatrix} i\alpha/|k| \\ 1 \end{pmatrix}, \]
(2.25)
where
\[ Y_{k,\alpha} = -i\alpha|k| \varphi_k^{in} + g_k^{in}. \]
(2.26)
For the next term, \( U_1 = (\psi_1, q_1)^t \), Eq. (2.4b) reads as
\[ \frac{\partial \psi_1}{\partial s} + q_1 = \Delta \psi_0 \]
(2.27a)
\[ \frac{\partial q_1}{\partial s} + \Delta q_1 = -\text{div}(q_0 \nabla \psi_0) \]
(2.27b)
then the system for the \( k \)-mode is
\[ \frac{d\psi_{1,k}}{ds} + q_{1,k} = -|k|^2 \psi_{0,k} \]
(2.28a)
\[
\frac{dq_{1,k}}{ds} - |k|^2 \psi_{1,k} = \sum_{j+l=k} (k \cdot l)q_{0,j} \psi_{0,l} \quad (2.28b)
\]

In particular \( k = 0 \), (2.28a, b) becomes
\[
\frac{d\psi_{1,0}}{ds} + q_{1,0} = 0 \quad \text{and} \quad \frac{dq_{1,0}}{ds} = 0 \quad (2.29)
\]

with zero initial data. For \( k \neq 0 \), by variation of parameter, we have
\[
\begin{pmatrix}
\psi_{1,k}(s) \\
\psi_{0,k}(s)
\end{pmatrix} = \int_0^s e^{-(s-r)R_k} \begin{pmatrix}
-|k|^2 \psi_{0,k} \\
\sum_{j+l=k} (k \cdot l)q_{0,j} \psi_{0,l}
\end{pmatrix} \, dr \quad (2.30)
\]

The integrand in (2.30) can be computed with the help of (2.25);
\[
e^{-R_k} \left( \frac{-|k|^2 \psi_{0,k}}{\sum_{j+l=k} (k \cdot l)q_{0,j} \psi_{0,l}} \right) = \left\{ \frac{1}{4} \sum_{\alpha, \gamma = \pm 1} e^{i(\alpha + \gamma)l|k|} \tau \alpha \gamma |k|^2 Y_{k,\alpha} \right. \\
+ \frac{1}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop j+l=k} i \beta (k \cdot l') e^{i(\alpha |j| + \beta |l| + \gamma |k|) \tau Y_{j,\alpha} Y_{l,\beta}} \left( \frac{-i \gamma / |k|}{1} \right) \right\} \quad (2.31)
\]

where \( l' = \frac{l}{|l|}, \quad |l|^2 = \sum_{n=1}^N l_n^2, \quad l \neq 0 \). The secular contribution of the linear and nonlinear terms on the right hand side of (2.31) are given respectively by
\[
\frac{1}{4} \sum_{\alpha, \gamma = -1} -|k|^2 Y_{k,\alpha} \left( \frac{-i \gamma / |k|}{1} \right) = -\frac{|k|^2}{2} \left( \frac{\varphi^{in}_k}{g^\alpha_k} \right) \quad (2.32)
\]
\[
\frac{1}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop \alpha |j| + \beta |l| + \gamma |k| = 0} i \beta (k \cdot l') Y_{j,\alpha} Y_{l,\beta} \left( \frac{-i \gamma / |k|}{1} \right) \quad (2.33)
\]

Therefore, using (2.26), we obtain the following expression
\[
\begin{pmatrix}
\psi_{1,k}(s) \\
\psi_{0,k}(s)
\end{pmatrix} = e^{-sR_k} \left\{ \left[ -\frac{|k|^2}{2} \left( \frac{\varphi^{in}_k}{g^\alpha_k} \right) \right] + \frac{i}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop \alpha |j| + \beta |l| + \gamma |k| = 0} \beta (k \cdot l') Y_{j,\alpha} Y_{l,\beta} \left( \frac{-i \gamma / |k|}{1} \right) \right\} \quad (2.34)
\]

where \( F^{(1)}_k(s, \gamma, \varphi^{in}, g^{in}) \) contains the oscillating terms and is given by
\[ \mathcal{F}_k^{(1)}(s, \gamma, \varphi^\text{in}, g^\text{in}) = i \sum_{\alpha, \gamma = 1} \Lambda(s, (\alpha + \gamma)|k|)|k|^2 Y_{k, \alpha} \]
\[ + \frac{1}{8} \sum_{\substack{j+l=k \\alpha, \beta, \gamma = \pm 1 \\alpha|j| + \beta|l| + \gamma|k| \neq 0}} \beta(k \cdot l') \Lambda(s, \alpha|j| + \beta|l| + \gamma|k|) Y_{j, \alpha} Y_{l, \beta} \]  
(2.35)

with \( \Lambda \) defined by
\[ \Lambda : \mathbb{R} \times \mathbb{R}^* \to \mathbb{C}, \quad \Lambda(s, r) = \frac{1 - e^{-rs}}{r}. \]  
(2.36)

Taking into account the expressions (2.25) and (2.34) we can write the naive expansion of \( U_\varepsilon = (\psi_\varepsilon, q_\varepsilon) \) in the Fourier space, for \( k \neq 0 \),
\[ \left( \begin{array}{c} \psi_{k, \varepsilon}(s) \\ q_{k, \varepsilon}(s) \end{array} \right) \simeq e^{-sR_k} \left( \begin{array}{c} \left( \begin{array}{c} \varphi_k^\text{in} \\ g_k^\text{in} \end{array} \right) \\ + \frac{i}{8} \sum_{\substack{j+l=k \\ \alpha, \beta, \gamma = \pm 1 \\ \alpha|j| + \beta|l| + \gamma|k| = 0}} \beta(k \cdot l') Y_{j, \alpha} Y_{l, \beta} \left( \begin{array}{c} -i \gamma |k| \\ 1 \end{array} \right) \end{array} \right) + O(\varepsilon^2) \]  
(2.37)

which is equivalent to (2.10). For the mode \( k = 0 \) (2.37) simply reduces to
\[ \psi_{k,0}(s) \simeq O(\varepsilon^2), \quad q_{k,0}(s) \simeq O(\varepsilon^2). \]  
(2.38)

As discussed in (2.11) − (2.14). Returning to the slow time scale (with \( s = t/\varepsilon \)) we derive the renormalization group equations in the Fourier space;
\[ \frac{d\Psi_k}{dt} + \frac{|k|^2}{2} \Psi_k - \frac{1}{8} \sum_{\substack{j+l=k \\ \alpha, \beta, \gamma = \pm 1 \\ \alpha|j| + \beta|l| + \gamma|k| = 0}} [(j \cdot l) \Psi_j \Psi_l - Q_j Q_l] = 0 \]  
(2.39a)
\[ \frac{dQ_k}{dt} + \frac{|k|^2}{2} Q_k - \frac{1}{8} \sum_{\substack{j+l=k \\ \alpha, \beta, \gamma = \pm 1 \\ \alpha|j| + \beta|l| + \gamma|k| = 0}} [(k \cdot j) \Psi_j Q_l + (k \cdot l) \Psi_l Q_j] = 0 \]  
(2.39b)

Moreover, we have \( \bar{\Psi}_k = \Psi_{-k}, Q_k = Q_{-k} \) and (\( \Psi_0, Q_0 \)) = 0. Here the bar denotes the complex conjugate. Recall that
\[ (\Psi, Q) = \sum_{k \in \mathbb{Z}^N} (\Psi_k, Q_k) \exp(ik \cdot x) \]  
(2.40)

then the renormalized equation (2.39a, b) can be written in the physical space as a reaction-diffusion system
\[ \frac{\partial \Psi}{\partial t} - \frac{1}{2} \Delta \Psi + B_1(\Psi, Q) = 0 \]  
(2.41a)
\[
\frac{\partial Q}{\partial t} - \frac{1}{2} \Delta Q + \text{div}\left( B_2(\Psi, Q) \right) = 0 \tag{2.41b}
\]

where \( \Psi \) and \( Q \) satisfy
\[
\int_{\Omega} \Psi(x, t) dx = \int_{\Omega} Q(x, t) dx = 0 \tag{2.42a}
\]
\[
\Psi_{|t=0} = \varphi_{\text{in}}, \quad Q_{|t=0} = g_{\text{in}} \tag{2.42b}
\]

The quadratic forms \( B_1 \) and \( B_2 \) are defined by the Fourier expansions
\[
B_1(\Psi, Q) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{8} \sum_{j+l=k} \sum_{\alpha, \beta, \gamma = \pm 1} \left( \nabla \Psi \right)_j \cdot \left( \nabla \Psi \right)_l + Q_j Q_l \right) \tag{2.43a}
\]
\[
B_2(\Psi, Q) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{8} \sum_{j+l=k} \sum_{\alpha, \beta, \gamma = \pm 1} \left( \nabla \Psi \right)_j \cdot \left( \nabla \Psi \right)_l + (\nabla \Psi)_j Q_l \right) \tag{2.43b}
\]

(2.1) **Remark**  We rewrite (2.37) as
\[
\begin{align*}
\begin{pmatrix} \psi_{t, k}(s) \\ q_{t, k}(s) \end{pmatrix} & \approx e^{-sR_k} \left\{ \begin{pmatrix} \varphi_{\text{in}}^k \\ g_{\text{in}}^k \end{pmatrix} + \varepsilon s \left[ \frac{|k|^2}{2} \begin{pmatrix} \varphi_{\text{in}}^k \\ g_{\text{in}}^k \end{pmatrix} \right. \\
+ \frac{1}{8} \sum_{j+l=k} \sum_{\alpha, \beta, \gamma = \pm 1} \begin{pmatrix} (j \cdot l) \varphi_{\text{in}}^j \varphi_{\text{in}}^l - g_{\text{in}}^j g_{\text{in}}^l \\ (k \cdot j) \varphi_{\text{in}}^j g_{\text{in}}^l + (k \cdot l) \varphi_{\text{in}}^j g_{\text{in}}^l \end{pmatrix} \right] \right\} + O(\varepsilon^2).
\end{align*}
\]  

For the case of well-prepared initial data, that is, \( \Delta \varphi_{\text{in}} = g_{\text{in}} = 0 \) or equivalently \( -|k|^2 \varphi_{\text{in}}^k = g_{\text{in}}^k = 0 \) for each \( k \), then it follows from (2.44) that \( Q = 0 \) (from the second component) and the renormalized equation for \( \Psi \) in the Fourier space is
\[
\frac{d\Psi_k}{dt} - \frac{1}{8} \sum_{j+l=k} \sum_{\alpha, \beta, \gamma = \pm 1} (j \cdot l) \Psi_j \Psi_l = 0. \tag{2.45}
\]

Therefore the renormalization group equations of the compressible potential flow equations (1.5) are identically zero for density fluctuation, \( Q = 0 \) and a Bernoulli equation for potential \( \Psi \):
\[
d\frac{\Psi}{dt} + B_1(\Psi, 0) = 0. \tag{2.46}
\]

§3 **Renormalization group equation for density fluctuation and velocity**
The same discussion as developed in the previous section also applied to (1.8)–(1.10). Introducing the time scale \( s = t/\varepsilon \) and setting
\[
v_\varepsilon(s) = u_\varepsilon(cs), \quad q_\varepsilon(s) = g_\varepsilon(cs),
\]
we can rewrite (1.8)–(1.10) in the weakly nonlinear problem for \( \tilde{U}_\varepsilon = (v_\varepsilon, q_\varepsilon)' \)
\[
\frac{d}{ds} \tilde{U}_\varepsilon + \mathcal{L} \tilde{U}_\varepsilon = \varepsilon \tilde{F}(\tilde{U}_\varepsilon), \quad \tilde{U}_\varepsilon(0) = \tilde{U}^\text{in} = \begin{pmatrix} u^\text{in} \\ g^\text{in} \end{pmatrix}.
\]
Let \( \tilde{U}_0 = (v_0, q_0)' \), then (2.4a) becomes
\[
\frac{d}{ds} v_0 + \nabla q_0 = 0, \quad \frac{d}{ds} q_0 + \text{div} v_0 = 0
\]
and thus
\[
\frac{\partial^2}{\partial s^2} q_0 - \Delta q_0 = 0, \quad \frac{\partial^2}{\partial s^2} v_0 - \nabla \times (\nabla \times v_0) - \Delta v_0 = 0.
\]
Similarly we introducing the Fourier expansion
\[
q_0 = \sum_k q_{0,k} e^{ikz}, \quad v_0 = \sum_k v_{0,k} e^{ikz},
\]
\[
u^\text{in} = \sum_k u^\text{in}_k e^{ikz}, \quad g^\text{in} = \sum_k g^\text{in}_k e^{ikz},
\]
then (3.3) become an infinite system of ordinary differential equations
\[
\frac{dv_0,k}{ds} + ikq_0,k = 0 \quad (3.6a)
\]
\[
\frac{dq_0,k}{ds} + ik \cdot v_{0,k} = 0. \quad (3.6b)
\]
We consider the matrix operator \( \mathcal{P}_k : \mathbb{C}^N \to \mathbb{C}^N, \mathcal{P}_k(z) = z - k'(k' \cdot z) \). Note that \( \sum_k \mathcal{P}_k v_{0,k} e^{ikz} = \mathcal{P} v_0 \), where \( \mathcal{P} \) is the Leray-Helmholtz projector onto the divergence-free space \( H \). For \( k = 0 \) we find
\[
q_{0,0} = g^\text{in}_{0,0}, \quad \text{and} \quad v_{0,0} = u^\text{in}_{0,0} = 0. \quad (3.7)
\]
Here we use the fact that \( \int_\Omega u^\text{in}(x)dx = \int_\Omega g^\text{in}(x)dx = 0 \). If \( k \neq 0 \) we obtain the following system:
\[
\frac{d}{ds} (\mathcal{P}_k v_0,k) = 0 \quad (3.8a)
\]
\[
\frac{d}{ds} (k' \cdot v_{0,k}) + i|k|q_0,k = 0 \quad (3.8b)
\]
\[
\frac{d}{ds} q_{0,k} + i|k|(k' \cdot v_{0,k}) = 0. \quad (3.8c)
\]
We set
\[
\hat{R}_k = \begin{pmatrix} 0 & i|k| \\ i|k| & 0 \end{pmatrix},
\]
then the associated exponential operator is given by
\[ e^{s \hat{R}_k} = \frac{1}{2} \sum_{\alpha=\pm 1} e^{i\alpha |k| s} \hat{R}_{k,\alpha}, \]

where
\[ \hat{R}_{k,\alpha} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \quad \alpha = 1, -1. \]

Thus, the solution of (3.8a, b, c), with initial condition \((u^i, g^i)^t\), is given explicitly as
\[ \mathcal{P}_k v_{0,k}(s) = \mathcal{P}^i_k u^i_k \]
\[ \begin{pmatrix} (k' \cdot v_{0,k}(s)) \\ q_{0,k}(s) \end{pmatrix} = \frac{1}{2} \sum_{\alpha=\pm 1} e^{i\alpha |k| s} \hat{Y}_{k,\alpha} \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \]

where
\[ \hat{Y}_{k,\alpha} = k' \cdot u^i_k - \alpha g^i_k. \]

For the next term, \(\hat{U}_1 = (v_1, q_1)^t\), (2.4b) becomes
\[ \frac{\partial v_1}{\partial s} + \nabla q_1 = \Delta v_0 \]
\[ \frac{\partial q_1}{\partial s} + \text{div}\, v_1 = -\text{div}(q_0 v_0) \]

and the associated system for the \(k\)-mode is
\[ \frac{d}{ds}(\mathcal{P}_k v_{1,k}) = -|k|^2(\mathcal{P}_k v_{0,k}) \]
\[ \frac{d}{ds}(k' \cdot v_{1,k}) + i|k| q_{1,k} = -|k|^2(k' \cdot v_{0,k}) \]
\[ \frac{d}{ds} q_{1,k} + i|k|(k' \cdot v_{1,k}) = -i \sum_{j+l=k} (k \cdot v_{0,l}) q_{0,j}. \]

We are only interested in the case when \(k \neq 0\). By variation of parameter, we have
\[ \begin{pmatrix} (k' \cdot v_{1,k}(s)) \\ q_{1,k}(s) \end{pmatrix} = \int_0^s e^{-(s-\tau) \hat{R}_k} \begin{pmatrix} -|k|^2(k' \cdot v_{0,k}) \\ -i \sum_{j+l=k} (k \cdot v_{0,l}) q_{0,j} \end{pmatrix} d\tau. \]

The integrand in (3.16) is decomposed into three parts with the help of (3.12)
\[ e^{s \hat{R}_k} \begin{pmatrix} -|k|^2(k' \cdot v_{0,k}) \\ -i \sum_{j+l=k} (k \cdot v_{0,l}) q_{0,j} \end{pmatrix} = \left\{ \frac{1}{4} \sum_{\alpha, \gamma = \pm 1} e^{i(\alpha + \gamma) k |\gamma| s} \hat{Y}_{k,\alpha} \right\} 
+ \frac{-i}{4} \sum_{\alpha, \gamma = \pm 1} e^{i|\gamma| s} (-\alpha \gamma)(j \cdot P lv_{0,l}) \hat{Y}_{j,\alpha} 
+ \frac{-i}{8} \sum_{\alpha, \beta, \gamma = \pm 1} e^{i|\beta| s} (-\alpha \gamma)(j \cdot P lv_{0,l}) \hat{Y}_{j,\alpha} \hat{Y}_{j,\beta}. \]
Using (3.13) and integrating, we obtain the following expression:

\[
\left( \frac{(k' \cdot v_{1,k})(s)}{q_{1,k}(s)} \right) = e^{-sR_k} \left\{ \begin{array}{l}
- \frac{|k|^2}{2} \left( \frac{(k' \cdot u_k^{in})}{g_k^{in}} \right) \\
+ \frac{-i}{2} \sum_{j+l=k \atop |j|=|k|} (j \cdot P_l u_l^{in}) \left( \frac{(j' \cdot u_j^{in})}{g_j^{in}} \right) \\
+ \frac{-i}{8} \sum_{j+l=k \atop \alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} \left[ \alpha \beta k^2 (j' \cdot u_j^{in})(l' \cdot u_l^{in}) + |k| g_j^{in} g_l^{in} \right] \\
+ \sum_{\gamma = \pm 1} F_k^{(1)}(s, \gamma, \varphi^{in}, g^{in}) \left( \frac{1}{\gamma} \right) \end{array} \right. \]

where \( F_k^{(1)} \) contains the oscillating terms and is given by

\[
F_k^{(1)}(s, \gamma, u^{in}, g^{in}) = -\frac{1}{4} \sum_{\gamma = \pm 1} \Lambda(s, (\alpha + \gamma)|k|) |k|^2 \tilde{Y}_{k, \alpha} \\
+ \frac{i}{4} \sum_{j+l=k \atop \alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} (j \cdot P_l v_0, l) \Lambda(s, |j| + \gamma|k|) \tilde{Y}_{j, \alpha} \\
+ \frac{-i}{8} \sum_{j+l=k \atop \alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} (\alpha \beta |k| (j' \cdot u_j^{in})(l' \cdot u_l^{in}) + |k| g_j^{in} g_l^{in}) \tilde{Y}_{j, \alpha} \tilde{Y}_{l, \beta} \\
\]

and \( \Lambda \) is defined the same as (2.36). Taking into account the expressions above the naive expansion of \( \tilde{U}_e \) in the Fourier space, for \( k \neq 0 \), is given by

\[
P_k v_{e,k}(s) = P_k u_k^{in} + \varepsilon s (|k|^2) P_k u_k^{in} \\
\]

\[
\left( \frac{(k' \cdot v_{e,k})(s)}{q_{e,k}(s)} \right) \approx e^{-sR_k} \left\{ \begin{array}{l}
- \frac{|k|^2}{2} \left( \frac{(k' \cdot u_k^{in})}{g_k^{in}} \right) + \varepsilon s \left[ -\frac{|k|^2}{2} \left( \frac{(k' \cdot u_k^{in})}{g_k^{in}} \right) \right] \\
+ \frac{-i}{2} \sum_{j+l=k \atop |j|=|k|} (j \cdot P_l u_l^{in}) \left( \frac{(j' \cdot u_j^{in})}{g_j^{in}} \right) \\
+ \frac{-i}{8} \sum_{j+l=k \atop \alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} \left( \alpha \beta |k| (j' \cdot u_j^{in})(l' \cdot u_l^{in}) + |k| g_j^{in} g_l^{in} \right) \\
+ \varepsilon \sum_{\gamma = \pm 1} F_k^{(1)}(s, \gamma, u^{in}, g^{in}) \left( \frac{1}{\gamma} \right) \right. + O(\varepsilon^2). \]

\]
For the mode \( k = 0 \) we simply have \( v_{k,0}(s) \approx O(\epsilon^2) \) and \( q_{k,0}(s) \approx O(\epsilon^2) \). In general, the renormalization group equations in the Fourier space are (with \( s = t/\epsilon \))

\[
\frac{d\xi_k}{dt} + |k|^2 \xi_k = 0, \quad k \cdot \xi_k = 0 \quad (3.21a)
\]

\[
\frac{d\psi_k}{dt} + \frac{|k|^2}{2} \psi_k + \frac{i}{2} \sum_{j+l=k \atop |j|=|k|} (j \cdot \xi_l) \psi_j 
+ \frac{i}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} \alpha \beta |k| \psi_j \psi_l + |k| Q_j Q_l = 0 \quad (3.21b)
\]

\[
\frac{dQ_k}{dt} + \frac{|k|^2}{2} Q_k + \frac{i}{2} \sum_{j+l=k \atop |j|=|k|} (j \cdot \xi_l) Q_j 
+ \frac{i}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} (k \cdot (\xi_j + j^l \psi_j)) Q_l + (k \cdot (\xi_l + l^j \psi_l)) Q_j = 0 \quad (3.21c)
\]

Moreover, we have \( \tilde{\xi}_k = \xi_{-k}, \tilde{\psi}_k = \psi_{-k}, \tilde{Q}_k = \bar{Q}_{-k}, \xi_0 = 0 \) (vector sense) and \( \psi_0 = 0 = Q_0 \). Here the bar denotes the complex conjugate. Defining

\[
\varpi = \sum_{k \in \mathbb{Z}^N} k' \psi_k \exp(ik \cdot x)
\]

then the renormalized equation \((3.21a, b, c)\) can be written in the physical space as a convection-diffusion system.

\[
\frac{\partial \xi}{\partial t} - \Delta \xi = 0, \quad \text{div} \xi = 0 \quad (3.23a)
\]

\[
\frac{\partial \varpi}{\partial t} - \frac{1}{2} \Delta \varpi + \nabla \hat{B}_1(\varpi, Q) + \nabla \hat{B}_3(\xi, \varpi) = 0 \quad (3.23b)
\]

\[
\frac{\partial Q}{\partial t} - \frac{1}{2} \Delta Q + \text{div} \hat{B}_2(\xi + \varpi, Q) + \text{div} \hat{B}_4(\xi, Q) = 0 \quad (3.23c)
\]

where \( \xi, \varpi \) and \( Q \) satisfy

\[
\int_{\Omega} \xi(x,t) dx = \int_{\Omega} \varpi(x,t) dx = 0, \quad \int_{\Omega} Q(x,t) dx = 0 \quad (3.24)
\]

\[
|\xi|_{t=0} = \mathcal{P} u_{\text{in}}, \quad |\varpi|_{t=0} = u_{\text{in}} - \mathcal{P} u_{\text{in}}, \quad |Q|_{t=0} = g_{\text{in}} \quad (3.25)
\]

and the quadratic forms \( \hat{B}_i, i = 1, 2, 3, 4 \) are defined by the Fourier expansions

\[
\hat{B}_1(\varpi, Q) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{8} \sum_{\alpha, \beta, \gamma = \pm 1 \atop |\alpha| + |\beta| + |\gamma| = 0} \alpha \beta |k| \varpi_j \varpi_l + Q_j Q_l \right) \quad (3.26a)
\]
\[ \hat{B}_3(\xi, \varpi) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{2} \sum_{|j| = |k|} \xi_j \cdot \varpi_j \right) \]  
\[ \hat{B}_4(\xi, Q) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{2} \sum_{|j| = |k|} \xi_j Q_j \right) \]  
\[ \hat{B}_2(\xi + \varpi, Q) = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{8} \sum_{\alpha, \beta, \gamma = \pm 1} \frac{1}{\alpha|\beta| + \beta|\alpha| + \gamma|\alpha| = 0} (\xi_j + \varpi_j) Q_l + (\xi_l + \varpi_l) Q_j \right) \]  

In particular, if the initial data \( \xi(x, 0) = \mathcal{P} u^\text{in} = 0 \), it follows immediately from the explicit form of the heat equation that \( \xi(x, t) = 0 \) for all \( x, t \) then \( \hat{B}_3 = \hat{B}_4 = 0 \) and \( \hat{B}_2(\xi + \varpi, Q) = \hat{B}_2(\varpi, Q) \) and the renormalization group equations (3.23b, c) are equivalent to (2.41a, b). Generally, for nontrivial initial data \( \xi(x, 0) = \mathcal{P} u^\text{in} \neq 0 \) we introduce the potential function \( \Phi \) satisfying

\[ \xi = \nabla \Phi_1, \quad \varpi = \nabla \Phi_2, \quad \Phi = \Phi_1 + \Phi_2 \]  

then the renormalization group equations (3.23a, b, c) become

\[ \frac{\partial \Phi_1}{\partial t} = h_1(x, t), \quad \Delta \Phi_1 = 0 \]  
\[ \frac{\partial \Phi_2}{\partial t} = -\frac{1}{2} \Delta \Phi_2 + \hat{B}_3(\nabla \Phi_1, \nabla \Phi_2) + \hat{B}_4(\nabla \Phi_1, Q) = h_2(x, t) \]  

where \( \nabla h_i = 0, i = 1, 2 \) or equivalently

\[ \frac{\partial \Phi}{\partial t} - \frac{1}{2} \Delta \Phi + \hat{B}_4(\nabla \Phi_2, Q) + \hat{B}_3(\nabla \Phi_1, \nabla \Phi_2) = h(x, t), \]  
\[ \frac{\partial Q}{\partial t} - \frac{1}{2} \Delta Q + \text{div} \hat{B}_2(\nabla \Phi, Q) + \text{div} \hat{B}_4(\nabla \Phi_1, Q) = 0. \]  

with \( \nabla h = 0 \).

We can also derive the renormalization group from equations (3.6a, b) directly. Set

\[ \tilde{L}_k = \begin{pmatrix} 0 & ik \cr i k^t & 0 \end{pmatrix} \]  

Then the matrix \( \tilde{L}_k \) has eigenvalues 0 and \( \pm i|k| \). The eigenvalue 0 is of multiplicity \( n - 1 \) with the eigenvectors \( e_j(k) = (\tilde{e}_j(k), 0)^t, \quad j = 1, 2, \ldots, n - 1 \),

\[ k \cdot \tilde{e}_j(k) = 0, \quad \tilde{e}_j(k) \cdot \tilde{e}_j(k) = \delta_{ij}. \]

The eigenvalues \( \pm i|k| \) have the eigenvectors \( e_\gamma(k) = (k/|k|, \gamma)^t \). Then the associated exponential operator has the following orthogonal decomposition
\[ e^{s \hat{L}_k} w_k = S_k w_k + \frac{1}{2} \sum_{\gamma = \pm 1} e^{i \gamma |k|^2 (w_k \cdot e_\gamma(k))} e_\gamma(k) \]  

(3.32)

with \( S_k = \text{diag}[\hat{P}_k, 0] \) satisfying

\[ \hat{P}_k(k) = 0, \quad \hat{P}_k(v_k) \cdot k = 0, \quad \text{and} \quad \hat{P}_k^2 = \hat{P}_k. \]

When \( N = 2 \) and \( N = 3 \), \( S_k \) is given respectively, by

\[ \frac{1}{|k|^2} \begin{pmatrix} k_2^2 & -k_1 k_2 & 0 \\ -k_1 k_2 & k_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{|k|^2} \begin{pmatrix} |k_{23}|^2 & -k_1 k_2 & -k_1 k_3 & 0 \\ -k_1 k_2 & |k_{13}|^2 & -k_2 k_3 & 0 \\ -k_1 k_3 & -k_2 k_3 & |k_{12}|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

(3.33)

Using the properties of \( \hat{L}_k \) and the similar calculations as above we obtain the naive expansion of \( \hat{U}_r \) in the Fourier space, for \( k \neq 0 \),

\[ \begin{pmatrix} v_{r,k}(s) \\ q_{r,k}(s) \end{pmatrix} \simeq e^{-s \hat{L}_k} \begin{pmatrix} u_{r,k}^n \\ g_{r,k}^n \end{pmatrix} + \epsilon s \begin{pmatrix} u_{r,k}^n \\ g_{r,k}^n \end{pmatrix} + \frac{|k|^2}{2} \begin{pmatrix} (k' \cdot u_{r,k}^n)(k' \cdot g_{r,k}^n) \\ (k' \cdot g_{r,k}^n)(k' \cdot u_{r,k}^n) \end{pmatrix} + \frac{i}{2} \sum_{j+l=|k|} \begin{pmatrix} (j' \cdot \hat{P}_j u_{r,j}^n)(k' \cdot g_{r,k}^n) \\ (j' \cdot \hat{P}_j g_{r,j}^n)(k' \cdot u_{r,k}^n) \end{pmatrix} \]

(3.34)

\[ + \frac{i}{8} \sum_{\gamma = \pm 1} \sum_{\alpha \beta \gamma = \pm 1} \begin{pmatrix} \alpha \beta |k|(j' \cdot u_{r,j}^n)(l' \cdot u_{r,l}^n)k' + |k|g_{r,j}^n g_{r,l}^n k' \\ (k \cdot u_{r,k}^n)g_{r,j}^n + (k \cdot u_{r,k}^n)g_{r,l}^n \end{pmatrix} + O(\epsilon^2). \]

For the mode \( k = 0 \) we simply have \( v_{r,0}(s) \simeq O(\epsilon^2) \) and \( q_{r,0}(s) \simeq O(\epsilon^2) \). For \( k \neq 0 \), the renormalization group equations in the Fourier space are (with \( s = t/\epsilon \))

\[ \frac{d \tilde{V}_k}{dt} + |k|^2 \hat{P}_k \tilde{V}_k + \frac{|k|^2}{2} (k' \cdot \tilde{V}_k) k' + \frac{i}{2} \sum_{j+l=|k|} (j' \cdot \hat{P}_j \tilde{V}_j)(j' \cdot \tilde{V}_j) k' \]

\[ + \frac{i}{8} \sum_{\gamma = \pm 1} \sum_{\alpha \beta \gamma = \pm 1} \begin{pmatrix} \alpha \beta |k|(j' \cdot \tilde{V}_j)(l' \cdot \tilde{V}_l) + |k|\hat{Q}_j \hat{Q}_l \end{pmatrix} k' = 0 \]

(3.35a)

\[ \frac{d \hat{Q}_k}{dt} + \frac{|k|^2}{2} \hat{Q}_k + \frac{i}{2} \sum_{j+l=|k|} (j' \cdot \hat{P}_j \tilde{V}_j) \hat{Q}_j \]

\[ + \frac{i}{8} \sum_{\gamma = \pm 1} \sum_{\alpha \beta \gamma = \pm 1} \begin{pmatrix} (k \cdot \tilde{V}_j) \hat{Q}_l + (k \cdot \tilde{V}_l) \hat{Q}_j \end{pmatrix} = 0. \]

(3.35b)
Moreover, we have $\tilde{V}_k = \hat{V}_{-k}$, $\tilde{Q}_k = \hat{Q}_{-k}$ and $(\tilde{V}_0, \tilde{Q}_0) = 0$. We also employ the notation

$$\langle \hat{V}, \hat{Q} \rangle = \sum_{k \in \mathbb{Z}^N} (\hat{V}_k, \hat{Q}_k) \exp(ik \cdot x)$$

(3.36)

then the renormalized equation (3.35a, b) can be written in the physical space as

$$\frac{\partial \hat{V}}{\partial t} - \frac{1}{2} \Delta (\hat{V} + \hat{\Omega}) + \nabla \hat{B}_3(\hat{\Omega}, \hat{\Omega}) + \nabla \hat{B}_1(\hat{\Omega}, \hat{Q}) = 0$$  

(3.37a)

$$\frac{\partial \hat{Q}}{\partial t} - \frac{1}{2} \Delta \hat{Q} + \text{div} \hat{B}_4(\hat{\Omega}, \hat{Q}) + \text{div} \hat{B}_2(\hat{V}, \hat{Q}) = 0$$  

(3.37b)

$$\int_{\Omega} \hat{V}(x, t) dx = 0, \quad \int_{\Omega} \hat{Q}(x, t) dx = 0$$  

(3.37c)

$$\hat{V}|_{t=0} = u^\text{in}, \quad \hat{Q}|_{t=0} = g^\text{in}$$  

(3.37d)

where $\hat{\xi}$ and $\hat{\omega}$ have the Fourier expansion

$$\hat{\xi} = \sum_{k \in \mathbb{Z}^N} \hat{\xi}_k \exp(ik \cdot x), \quad \hat{\omega} = \sum_{k \in \mathbb{Z}^N} (k' \cdot \hat{V}_k) k' \exp(ik \cdot x)$$

with properties

$$\hat{V} = \hat{\xi} + \hat{\omega}, \quad \text{div} \hat{\xi} = 0.$$

It is clear that this renormalization group is equivalent to (3.23a-c).

§4 Properties of the renormalization group

In this section we will prove the global existence of weak solutions to the renormalization group equation (2.41a, b). To avoid complications at the boundary, we concentrate below on the case where $x \in \mathbb{T}^N$, the $N$-dimensional torus. That is, we will study the functions in the periodic case. First we have the orthogonal property of the nonlinear terms.

(4.1) Lemma  We have the following orthogonality property of the nonlinear terms

$$\left< B_1(\Psi, Q) \big| \Delta \Psi \right> + \left< B_2(\Psi, Q) \big| \nabla Q \right> = 0$$  

(4.1)

Proof: For the sake of simplification we denote $\sum_{(j,l,k)}$ as summation over the indices $(\alpha, \beta, \gamma)$ and $(j, l, k)$ satisfying $j + l = k$, $\alpha, \beta, \gamma = \pm 1$, and $\alpha |j| + \beta |l| + \gamma |k| = 0$, i.e.,

$$\sum_{(j,l,k)} \equiv \sum_{\alpha, \beta, \gamma = \pm 1} \sum_{\alpha |j| + \beta |l| + \gamma |k| = 0}$$

Using the definitions of $B_1$ and $B_2$ given by (2.43a, b), we have
\[
\langle B_1(\Psi, Q) | \Delta \Psi \rangle + \langle B_2(\Psi, Q) | \nabla Q \rangle \\
= \frac{1}{8} \sum_{(j,l,k)} \left[-(j \cdot l) \Psi_j \Psi_l Q_j Q_l \right] (-|k|^2 \Psi_k) + \left( ij \Psi_j Q_l + il \Psi_l Q_j \right) \cdot (-i k \hat{Q}_k) \right]
\]
\[
= \frac{1}{8} \sum_{(j,l,-k)} \left[ (|k|^2) \Psi_j \Psi_l Q_j Q_l - |k|^2 \Psi_k Q_j Q_l \right] \\
- (k \cdot j) \Psi_j Q_l \hat{Q}_k - (k \cdot l) \Psi_l Q_k \hat{Q}_k \right]
\]
\[
= \frac{1}{8} \sum_{(j,l,-k)} \left[ \frac{1}{3} \left( |k|^2 (j \cdot l) + |j|^2 (l \cdot k) + |l|^2 (k \cdot j) \right) \Psi_j \Psi_l \Psi_k \\
- \left( |k|^2 + (k \cdot j) + (k \cdot l) \right) \Psi_k Q_l \right] \\
= \frac{1}{8} \sum_{(j,l,-k)} \left[ \frac{1}{3} \left( \alpha \beta \gamma \right) |j||l||k| (|k|^2) \right. \\
- k \cdot (j + l + k) \Psi_k Q_j Q_l \right] \\
= 0.
\]

This proves the lemma.

Due to the orthogonal property of the bilinear form (4.1) we have the conservation of energy;
\[
\frac{d}{dt} \left( \int_\Omega |\nabla \Psi|^2 + |Q|^2 dx \right) + \int_\Omega |\Delta \Psi|^2 + |\nabla Q|^2 dx = 0,
\]
which follows immediately by multiplying (2.41a) by \( \Delta \Psi \) and (2.41b) by \( Q \) and using the orthogonal property above. Then the classical proofs of ([20],[21],[26]) can then be adapted to the existence of weak solutions of the renormalization group equation (2.41a,b). The following theorem then establishes global existence of weak solutions to the problem.

**Theorem** (4.2) *Given \((\psi^{in}, q^{in}) \in \dot{H}^1 \times \dot{L}^2\), there exists a unique pair of functions*
\[
(\Psi, Q) \in C\left([0, \infty); \dot{H}^1 \times \dot{L}^2\right) \cap L^2_{loc}\left([0, \infty); \dot{H}^2 \times \dot{H}^1\right)
\]
*satisfies the weak form of the system (2.41a,b) given respectively by*
\[
(\nabla \phi, \nabla Q(t_2)) - (\nabla \phi, \nabla Q(t_1)) + \frac{1}{2} \int_{t_1}^{t_2} (\Delta \phi, \Delta \Psi) dt - \int_{t_1}^{t_2} (\Delta \phi, B_1(\Psi, Q)) dt = 0
\]
\[
(\psi, Q(t_2)) - (\psi, Q(t_1)) + \frac{1}{2} \int_{t_1}^{t_2} (\nabla \psi, \nabla Q) dt - \int_{t_1}^{t_2} (\nabla \psi, B_2(\Psi, Q)) dt = 0
\]
*for every \([t_1, t_2] \in [0, \infty)\), and every test functions \(\phi, \psi \in C^\infty_c(\Omega)\). The solution satisfies the energy relation.*
\[
\int_{\Omega} |\nabla \Psi|^2 + |Q|^2 \, dx + \int_0^t \int_{\Omega} |\Delta \Psi|^2 + |\nabla Q|^2 \, dx \, dt \leq \int_{\Omega} |\nabla \varphi^m|^2 + |g^m|^2 \, dx \quad (4.5)
\]
for every \( t \in [0, \infty) \). Furthermore,
\[
(\Psi, Q) \in C([0, \infty); \dot{H}^1 \times L^2) .
\]  
(4.6)
Moreover, if \((\varphi^m, g^m) \in H^s \times H^{s-1}\), one has
\[
(\Psi, Q) \in C([0, \infty); w-(\dot{H}^s \times \dot{H}^{s-1})) \cap L^2_{loc}([0, \infty); \dot{H}^{s+1} \times \dot{H}^s)
\]  
(4.7)
Hence,
\[
(\Psi, Q) \in C([0, \infty); \dot{H}^s \times \dot{H}^{s-1}) .
\]  
(4.8)
In particular, when \( s > N/2 + 1 \), \((\Psi, Q) \in C([0, \infty) \times \Omega)\) by Sobolev embedding theorem. Here \( \dot{H}^s \) is the homogeneous Sobolev space of order \( s \) with zero mean
\[
\dot{H}^s = \left\{ u \in H^s \mid \int_{\Omega} u(x) \, dx = 0 \right\} , \quad s \geq 0 .
\]  
(4.9)

The idea of the proof of (4.2) Theorem is similar in spirit to that of Leray for the incompressible Navier-Stokes equations. The first step is to construct a family of approximate solutions \((\Psi_n, Q_n)\) constructed by any method that yields a consistent weak formulation and an energy relation — for example, the Fourier-Galerkin method. Let \( n \) be a positive integer. Let \( P_n \) denote the \( L^2 \)-orthogonal projection onto the span of the lowest \( n \) eigenvalues then apply \( P_n \) to (2.41a, b) we derive the Galerkin system of order \( n \)

\[
\frac{\partial \Psi_n}{\partial t} - \frac{1}{2} \Delta \Psi_n + P_n B_1(\Psi_n, Q_n) = 0
\]  
(4.10a)

\[
\frac{\partial Q_n}{\partial t} - \frac{1}{2} \Delta Q_n + P_n \text{div} \left( B_2(\Psi_n, Q_n) \right) = 0
\]  
(4.10b)

\[
(\Psi_n(x, 0), Q_n(x, 0)) = (P_n \varphi^m(x), P_n g^m(x))
\]  
(4.10c)

It can be shown that the \((\Psi_n, Q_n)\) satisfy the weak form of the regularized system \((4.10a, b)\)

\[
\langle \nabla \phi, \nabla \Psi_n(t_2) \rangle - \langle \nabla \phi, \nabla \Psi_n(t_1) \rangle + \frac{1}{2} \int_{t_1}^{t_2} \langle \Delta \phi, \Delta \Psi_n \rangle \, dt = 0
\]  
(4.11a)

\[
\langle \psi, Q_n(t_2) \rangle - \langle \psi, Q_n(t_1) \rangle + \frac{1}{2} \int_{t_1}^{t_2} \langle \nabla \psi, \nabla Q_n \rangle \, dt
\]
\[
- \int_{t_1}^{t_2} \langle \nabla \psi, P_n B_2(\Psi_n, Q_n) \rangle \, dt = 0
\]  
(4.11b)

for every \([t_1, t_2] \in [0, \infty)\), \( \phi, \psi \in C^\infty_c(\Omega) \) and the energy relation
\[ \| \Psi_n(t_2) \|_{H^1}^2 + \| Q_n(t_2) \|_{L^2}^2 + \int_{t_1}^{t_2} \| \Psi_n \|_{H^2}^2 + \| Q_n \|_{H^1}^2 \, dt = \| \Psi_n(t_1) \|_{H^1}^2 + \| Q_n(t_1) \|_{L^2}^2. \] (4.12)

The nonlinearities in (4.10a, b) are quadratic polynomials, hence locally Lipschitz, so that local existence of the regularized solutions \((\Psi_n, Q_n)\) is guaranteed by the Picard-Lindelöf existence theorem. The global energy relation (4.12) then assures that \((\Psi_n, Q_n)\) is in fact a global solution. The argument is standard so that we omit the details.

Applying the equicontinuity argument as given in [19], we deduce from the energy relation (4.12) that

\(\{ \Psi_n \}\) is a relatively compact set in \(C([0,\infty), w-H^1) \cap w-L^2_{loc}([0,\infty), w-H^2)\)

\(\{ Q_n \}\) is a relatively compact set in \(C([0,\infty), w-L^2) \cap w-L^2_{loc}([0,\infty), w-H^1)\)

Moreover, from the fact that the embedding

\(C([0,\infty), w-H^s-1) \cap w-L^2_{loc}([0,\infty), w-H^s) \hookrightarrow L^2_{loc}([0,\infty), H^s-1)\) (4.13)

is continuous, we can show that

\(\{ \Psi_n \}\) is a relatively compact set in \(L^2_{loc}([0,\infty), H^1)\) (4.14a)

\(\{ Q_n \}\) is a relatively compact set in \(L^2_{loc}([0,\infty), H^1)\) (4.14b)

endowed with the strong topologies. Thus there exist subsequences of \(\{ \Psi_n \}\) and \(\{ Q_n \}\), still denoted by \(\{ \Psi_n \}\) and \(\{ Q_n \}\), that converge respectively to \(\Psi\) and \(Q\) satisfying

\[ \Psi \in C([0,\infty), w-H^1) \cap L^2_{loc}([0,\infty), H^2) \] (4.15a)

\[ Q \in C([0,\infty), w-L^2) \cap L^2_{loc}([0,\infty), H^1) \] (4.15b)

in the following sense:

\[ \Psi_n \to \Psi \quad \text{in} \quad C([0,\infty), w-H^1), \] (4.16a)

\[ \Psi_n \to \Psi \quad \text{in} \quad w-L^2_{loc}([0,\infty), w-H^2), \] (4.16b)

\[ \Psi_n \to \Psi \quad \text{in} \quad L^2_{loc}([0,\infty), H^1), \] (4.16c)

\[ Q_n \to Q \quad \text{in} \quad C([0,\infty), w-L^2), \] (4.17a)

\[ Q_n \to Q \quad \text{in} \quad w-L^2_{loc}([0,\infty), w-H^1), \] (4.17b)

\[ Q_n \to Q \quad \text{in} \quad L^2_{loc}([0,\infty), L^2). \] (4.17c)

These compactness results are good enough to imply the convergence of the linear terms of (4.11a, b). In order to pass the limit of nonlinear term it
is sufficient to show the compactness of the nonlinear term \( \{ Q_n \nabla \Psi_n \} \). We deduce from (4.14a) and the compactness of injection \( H^2 \hookrightarrow H^1 \) that

\[
\{ \nabla \Psi_n \} \text{ is a relatively compact set in } L^2_{loc}([0, \infty), \dot{L}^2).
\tag{4.18}
\]

Employing (4.17c) and (4.18) we can conclude

\[
Q_n \nabla \Psi_n \to Q \nabla \Psi \quad \text{in } L^1_{loc}([0, \infty), \dot{L}^1).
\tag{4.19}
\]

Thus it is enough to pass to limit as \( n \to \infty \) in the nonlinear terms in (4.11a,b). Therefore \((\Psi, Q)\) satisfies the weak form (4.4a,b). From the fact that \( P_n(\varphi^{in}) \to \varphi^{in} \) in \( H^1 \), \( P_n(g^{in}) \to g^{in} \) in \( \dot{L}^2 \) and the convergences (4.16) - (4.17) we have

\[
\| \Psi(t) \|_{H^1} \leq \liminf_{n \to \infty} \| \Psi_n(t) \|_{H^1} \quad \text{and} \quad \| Q(t) \|_{\dot{L}^2} \leq \liminf_{n \to \infty} \| Q_n(t) \|_{\dot{L}^2}
\]

then the convergence of the energy relation (4.5) follows immediately by the Lebesgue's Dominated convergent theorem. Next, we prove (4.6). From the energy relation and the interpolation theorem we have

\[
\Psi \in L^2_{loc}([0, \infty); \dot{H}^2) \cap L^\infty([0, \infty); \dot{H}^1) \hookrightarrow L^2_{loc}([0, \infty); \dot{H}^{2-\alpha}) \quad (4.20a)
\]

\[
Q \in L^2_{loc}([0, \infty); \dot{H}^1) \cap L^\infty([0, \infty); \dot{L}^2) \hookrightarrow L^2_{loc}([0, \infty); \dot{H}^{1-\alpha}) \quad (4.20b)
\]

for \( 0 < \alpha < 1 \). Applying the Hölder inequality we have

\[
\dot{H}^{1-\alpha} \hookrightarrow \dot{W}^{1-\alpha, r}, \quad 1 < r = \frac{N}{N-1+\alpha} < \frac{N}{N-1} \quad (4.21)
\]

We deduce from (4.20a,b) (4.21) that

\[
B_1, B_2 \in L^1_{loc}([0, \infty); \dot{W}^{1-\alpha, r}). \quad (4.22)
\]

Thus the weak form (4.4a,b) implies the existence of a distributional time derivatives

\[
\partial_t \Psi \in L^2_{loc}([0, \infty); \dot{H}^{-\alpha}) + L^1_{loc}([0, \infty); \dot{W}^{1-\alpha, r}) \hookrightarrow L^\frac{1}{r}_{loc}([0, \infty); \dot{W}^{-\alpha, r})
\]

\[
\partial_t Q \in L^2_{loc}([0, \infty); \dot{H}^{-1-\alpha}) + L^1_{loc}([0, \infty); \dot{W}^{1-\alpha, r}) \hookrightarrow L^\frac{1}{r}_{loc}([0, \infty); \dot{W}^{-1-\alpha, r})
\]

Since the embedding

\[
\{ \Psi \in L^2_{loc}([0, \infty); \dot{H}^{2-\alpha}); \partial_t \Psi \in L^\frac{1}{r}_{loc}([0, \infty); \dot{W}^{-\alpha, r}) \} \hookrightarrow C([0, \infty); \dot{H}^{2-\alpha})
\tag{4.23a}
\]

\[
\{ Q \in L^2_{loc}([0, \infty); \dot{H}^{1-\alpha}); \partial_t Q \in L^\frac{1}{r}_{loc}([0, \infty); \dot{W}^{-1-\alpha, r}) \} \hookrightarrow C([0, \infty); \dot{H}^{1-\alpha})
\tag{4.23b}
\]

is continuous. This proves (4.6).

To show the uniqueness we assume that \((\Psi_1, Q_1)\) and \((\Psi_2, Q_2)\) are two solutions to equation (2.41), (2.42) and set \( \Psi = \Psi_1 - \Psi_2, Q = Q_1 - Q_2 \). Then
\[
(\Psi, Q) \text{ satisfies}
\]
\[
\frac{\partial \Psi}{\partial t} - \frac{1}{2} \Delta \Psi + [B_1(\Psi_1, Q_1) - B_1(\Psi_2, Q_2)] = 0 \quad (4.24a)
\]
\[
\frac{\partial Q}{\partial t} - \frac{1}{2} \Delta Q + \text{div}(B_2(\Psi_1, Q_1)) - \text{div}(B_2(\Psi_2, Q_2)) = 0 \quad (4.24b)
\]
\[
\Psi^{in} = 0, \quad Q^{in} = 0. \quad (4.24c)
\]

From the definition of \(B_1\) and \(B_2\), direct computation yields
\[
\left\langle B_1(\Psi_1, Q_1) - B_1(\Psi_2, Q_2) \big| \Delta \Psi \right\rangle + \left\langle B_2(\Psi_1, Q_1) - B_2(\Psi_2, Q_2) \big| \nabla Q \right\rangle
\]
\[
= \frac{1}{8} \sum_{(j,l,k)} \left\{ (-j \cdot l)(\Psi_{j,l} \Psi_{j,l} - \Psi_{2,j} \Psi_{2,l}) + (Q_{j,l} Q_{j,l} - Q_{2,j} Q_{2,l}) \right\} (-|k|^2 \tilde{\Psi}_k)
\]
\[
+ \left\{ ij(\Psi_{j,l} \Psi_{1,j} - \Psi_{2,j} \Psi_{2,l}) + il(\Psi_{1,l} \Psi_{1,l} - \Psi_{2,l} \Psi_{2,l}) \right\} (-i k \tilde{Q}_k)
\]
\[
= \frac{1}{8} \sum_{(j,l,k)} \left\{ (-j \cdot l)(\Psi_{j,l} \Psi_{j,l} + \Psi_{2,j} \Psi_{2,l} + \Psi_{2,l} \Psi_{j,l}) (-|k|^2 \tilde{\Psi}_k)
\]
\[
+ (Q_{j,l} Q_{j,l} + Q_{2,l} Q_{2,l} + Q_{2,l} Q_{j,l}) (-|k|^2 \tilde{\Psi}_k)
\]
\[
+ ij(\Psi_{j,l} Q_{j,l} + \Psi_{2,l} Q_{2,l} + \Psi_{j,l} Q_{2,l}) (-i k \tilde{Q}_k)
\]
\[
+ il(\Psi_{j,l} Q_{j,l} + \Psi_{2,l} Q_{2,l} + \Psi_{2,l} Q_{j,l}) (-i k \tilde{Q}_k)
\]
\[
= \frac{1}{8} \sum_{(j,l,k)} \left\{ (-j \cdot l)(\Psi_{j,l} \Psi_{j,l} + \Psi_{2,l} \Psi_{j,l} + \Psi_{j,l} \Psi_{2,l}) - |k|^2(\Psi_{2,j} Q_{j,l} + \Psi_{j,l} Q_{2,j})
\]
\[
- (k \cdot j)(\Psi_{2,j} Q_{1,j} + \Psi_{j,l} Q_{2,l} + \Psi_{2,l} Q_{2,l}) - (k \cdot l)(\Psi_{2,l} Q_{1,l} + \Psi_{l,j} Q_{2,j} + \Psi_{l,j} Q_{2,j})
\]
\[
= \frac{1}{8} \sum_{(j,l,k)} \left\{ (-j \cdot l)(\Psi_{2,j} Q_{j,l} + \Psi_{j,l} Q_{2,l}) + 2(k \cdot j) Q_{2,j} Q_{j,l} \Psi_{j,l} + |j|^2 \Psi_{2,j} Q_{l} Q_{k}
\]
\[
= \left\langle B_1(\Psi, Q) \big| \Delta \Psi \right\rangle + \left\langle B_2(\Psi, Q) \big| \nabla Q \right\rangle
\]

Multiplying (4.24a) by \(\Delta \Psi\) and (4.24b) by \(Q\), using the above identity and the Hölder inequality we derive the energy inequality
\[
\frac{d}{dt} \left\langle \|\Psi\|^2_{\dot{H}^1} + \|Q\|^2_{\dot{L}^2} \right\rangle + \|\Psi\|^2_{\dot{H}^2} + \|Q\|^2_{\dot{H}^1} \leq C \left( \|\Psi_2\|^2_{\dot{H}^1} + \|Q_2\|^2_{\dot{H}^1} + 2 \right) \left( \|\Psi\|^2_{\dot{H}^1} + \|Q\|^2_{\dot{L}^2} \right). \quad (4.25)
\]

It follows from the Gronwall's inequality that
\[
\|\Psi(t)\|^2_{\dot{H}^1} + \|Q(t)\|^2_{\dot{L}^2} \leq C \left( \|\Psi(0)\|^2_{\dot{H}^1} + \|Q(0)\|^2_{\dot{L}^2} \right)
\times \exp \left[ \int_0^T \left( \|\Psi_2\|^2_{\dot{H}^1} + \|Q_2\|^2_{\dot{H}^1} + 2 \right) dt \right]. \quad (4.26)
\]

Since \((\Psi_2, Q_2) \in L^2_{loc}(0, \infty); \dot{H}^2 \times \dot{H}^1\), the exponential factor is finite, hence the difference \((\Psi, Q)\) between two solutions remains zero if it is so initially.
(4.3) **Remark**  Actually, from (4.23) we have a better result than (4.6);

\[(\Psi, Q) \in C([0, \infty); \dot{H}^{2-\alpha} \times \dot{H}^{1-\alpha}), \quad 0 < \alpha < 1.\]

We also have similar result for (3.23a – c) or (3.37a – d). The argument is almost the same as above we therefore omit the detail.

(4.4) **Theorem**  Given initial data \((\varphi^{in}, g^{in}) \in \left(\dot{L}^2(\Omega)\right)^2\), there exist a unique triple of functions

\[(\xi, \omega, Q) \in C\left([0, \infty); w-(\dot{L}^2(\Omega))^3\right) \cap L_{loc}^2\left([0, \infty); (\dot{H}^4(\Omega))^3\right) \tag{4.27}\]

satisfies the weak form of the system (3.23a, b, c) given respectively by

\[
\begin{align*}
\langle \varphi, \xi(t_2) \rangle - & \langle \varphi, \xi(t_1) \rangle + \int_{t_1}^{t_2} \langle \nabla \varphi, \nabla \xi \rangle \, dt = 0, \\
\langle \phi, \omega(t_2) \rangle - & \langle \phi, \omega(t_1) \rangle + \frac{1}{2} \int_{t_1}^{t_2} \langle \nabla \phi, \nabla \omega \rangle \, dt - \int_{t_1}^{t_2} \left\langle \text{div} \phi, \dot{B}_1(\omega, Q) \right\rangle \, dt \\
& - \int_{t_1}^{t_2} \left\langle \text{div} \phi, \dot{B}_3(\xi, Q) \right\rangle \, dt = 0 \tag{4.28b} \\
\langle \psi, Q(t_2) \rangle - & \langle \psi, Q(t_1) \rangle + \frac{1}{2} \int_{t_1}^{t_2} \langle \nabla \psi, \nabla Q \rangle \, dt - \int_{t_1}^{t_2} \left\langle \nabla \psi, \dot{B}_2(\xi + \omega, Q) \right\rangle \, dt \\
& - \int_{t_1}^{t_2} \left\langle \nabla \psi, \dot{B}_4(\xi, Q) \right\rangle \, dt = 0 \tag{4.28c}
\end{align*}
\]

for every \([t_1, t_2] \in [0, \infty),\) and every test functions \(\varphi, \phi, \psi \in C_c^\infty(\Omega)\). The solution satisfies the energy relation

\[
\int_\Omega |\xi|^2 + |\omega|^2 + |Q|^2\, dx + \int_0^t \int_\Omega 2|\nabla \xi|^2 + |\nabla \omega|^2 + |\nabla Q|^2\, dx \, dt \\
\leq C \int_\Omega |\varphi^{in}|^2 + |g^{in}|^2 \, dx \tag{4.29}
\]

for every \(t \in [0, \infty).\) Furthermore,

\[(\xi, \omega, Q) \in C\left([0, \infty); (\dot{L}^2(\Omega))^3\right).\]

Moreover, if \((\varphi^{in}, g^{in}) \in \left(\dot{H}^s(\Omega)\right)^2,\) one has

\[
(\xi, \omega, Q) \in C\left([0, \infty); w-(\dot{H}^s(\Omega))^3\right) \cap L_{loc}^2\left([0, \infty); (\dot{H}^{s+1}(\Omega))^3\right). \tag{4.30}
\]

Hence,

\[(\xi, \omega, Q) \in C\left([0, \infty); (\dot{H}^s(\Omega))^3\right). \tag{4.31}\]

In particular, when \(s > N/2,\) \((\xi, \omega, Q) \in C([0, \infty) \times \Omega)\) by Sobolev embedding theorem.
§5 Comparison of the exact solution and its approximation

In this section, we will compare the exact solution $w_\varepsilon$ of (1.5) and its approximation $\overline{U}_\varepsilon$. It is clear from (2.15, 16) that we need to estimate the error term $R_\varepsilon$. Thus we need to treat the oscillation term $F_{osc}$ first. Employing the isometric property of the wave group $C(t)$ we deduce that the approximate solution $\overline{U}_\varepsilon$ given by (2.13, 14) will have the same regularity of $F_{osc}$. Indeed, from the Fourier expression of $F_{osc}$ in (2.37):

$$F_{osc,k} = \sum_{\gamma = \pm 1} F^{(1)}_k(s, \gamma, \Psi, Q) \left( -i\gamma/k \right),$$

(5.1)

where $F^{(1)}_k(s, \gamma, \Psi, Q)$ are defined by (2.35, 36). For $m > N/2$, the first component of $F_{osc}$ can be estimated by

$$\|F^{(1)}_{osc}\|_{\dot{H}^m} \leq C \left\{ \sum_k (|k||\Psi_k| + |Q_k|)^2 (1 + |k|^2)^m + \sum_k \left[ \sum_{j+l=k} (|j| + |l|) |j||l||j\Psi_j| + |Q_j||l\Psi_l| + |Q_l|) \right]^2 (1 + |k|^2)^m \right\}^{1/2} \leq C \left\{ \| (\Psi, Q) \|_{\dot{H}^{m+1} \times \dot{H}^m} + \| (\Psi, Q) \|_{\dot{H}^{m+3} \times \dot{H}^{m+2}} \right\}$$

Similarly, for second component and $m + 1 > N/2$ we have

$$\|F^{(2)}_{osc}\|_{\dot{H}^m} \leq C \left\{ \sum_k (|k||\Psi_k| + |Q_k|)^2 (1 + |k|^2)^{m+1} + \sum_k \left[ \sum_{j+l=k} (|j| + |l|) |j||l||j\Psi_j| + |Q_j||l\Psi_l| + |Q_l|) \right]^2 (1 + |k|^2)^{m+1} \right\}^{1/2} \leq C \left\{ \| (\Psi, Q) \|_{\dot{H}^{m+2} \times \dot{H}^{m+1}} + \| (\Psi, Q) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}} \right\}$$

Combining the above two inequalities we have for $m + 1 > N/2$

$$\|F_{osc}\|_{\dot{H}^{m+1} \times \dot{H}^m} \leq C \left\{ \| (\Psi, Q) \|_{\dot{H}^{m+3} \times \dot{H}^{m+2}} + \| (\Psi, Q) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}} \right\}.$$  

(5.2)

Furthermore from (4.5), we have for all $t$

$$\| (\Psi, Q)(t) \|_{\dot{H}^{m+1} \times \dot{H}^m}^2 + \int_0^t \| (\Psi, Q) \|_{\dot{H}^{m+2} \times \dot{H}^{m+1}}^2 \, dt \leq \| (\varphi^0, g^0) \|_{\dot{H}^{m+1} \times \dot{H}^m}^2$$

Taking the square of (5.2) using the above inequality and integrating over $[0, T]$ we derive
\[
\int_0^T \| \mathcal{F}_{osc} \|_{\dot{H}^{m+2} \times \dot{H}^{m+1}} dt \\
\leq C \left\{ \| (\varphi^i, g^i) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}}^2 + \| (\varphi^i, g^i) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}}^4 \right\}.
\]

Similarly we also have
\[
\| \mathcal{F}_{osc} \|_{\dot{H}^{m+1} \times \dot{H}^{m}} \leq C \left\{ \| (\varphi^i, g^i) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}}^2 + \| (\varphi^i, g^i) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}}^2 \right\}.
\]

Therefore we have proved the following proposition.

(5.1) Proposition  Let \( N < 4 \) and \( m \geq 1 \). Given
\[
W = (\Psi, Q) \in C([0, \infty); \dot{H}^{m+4} \times \dot{H}^{m+3}) \cap L^2_{loc}(0, \infty); \dot{H}^{m+5} \times \dot{H}^{m+4})
\]
solves the renormalization group equation (2.41) then the nonsecular term \( \mathcal{F}_{osc}(t/\epsilon, W(t)) \) satisfies
\[
\int_0^T \| \mathcal{F}_{osc}(t/\epsilon, W(t)) \|_{\dot{H}^{m+2} \times \dot{H}^{m+1}} dt \leq C(\varphi^i, g^i),
\]
\[
\sup_{t \in [0, T]} \| \mathcal{F}_{osc}(t/\epsilon, W(t)) \|_{\dot{H}^{m+1} \times \dot{H}^{m}} dt \leq \tilde{C}(\varphi^i, g^i)
\]
where \( C(\cdot) \) and \( \tilde{C}(\cdot) \) are constants independent of \( \epsilon \).

Now we estimate the error term \( \epsilon R_\epsilon \) given by (2.16)
\[
\epsilon R_\epsilon = \mathcal{F}(L(t/\epsilon)W(t)) - \mathcal{F}(U_\epsilon) + \epsilon L(t/\epsilon) \left( \nabla_W \mathcal{F}_{osc} \mathcal{F}_R(W) \right)
\]
\[
= - \mathcal{F}(\epsilon L(t/\epsilon) \mathcal{F}_{osc}) - Q(L(t/\epsilon)W, \epsilon L(t/\epsilon) \mathcal{F}_{osc})
\]
\[
- Q(\epsilon L(t/\epsilon) \mathcal{F}_{osc}, L(t/\epsilon)W) + \epsilon L(t/\epsilon) \left( \nabla_W \mathcal{F}_{osc} \mathcal{F}_R(W) \right)
\]
where the nonlinear operator \( Q \) is defined by
\[
Q \left( \begin{pmatrix} \psi_1 \\ q_1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ q_2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -\text{div}(q_2 \nabla \psi_1) \end{pmatrix}.
\]

We start from the nonlinear term \( \nabla_W \mathcal{F}_{osc}(t/\epsilon, W(t)) : \mathcal{F}_R(W) \). Applying the same arguments as in (5.2) to the explicit form of \( \mathcal{F}_{osc} \) we have
\[
\| \nabla_W \mathcal{F}_{osc}(t/\epsilon, W(t)) : \mathcal{F}_R(W) \|_{\dot{H}^{m+1} \times \dot{H}^{m}}
\]
\[
\leq C \left\{ \| \Psi_l \|_{\dot{H}^{m+2}} + \| (\nabla^3 \Psi + \nabla^2 Q) \cdot \nabla^2 \Psi_t \|_{\dot{H}^{m+1}} + \| (\nabla^2 \Psi + \nabla Q) \cdot \nabla^3 \Psi_t \|_{\dot{H}^{m+1}} \\
+ \| Q_l \|_{\dot{H}^{m+1}} + \| (\nabla^3 \Psi + \nabla^2 Q) \cdot \nabla Q_t \|_{\dot{H}^{m+1}} + \| (\nabla^2 \Psi + \nabla Q) \cdot \nabla^2 Q_t \|_{\dot{H}^{m+1}} \right\}
\]
\[
\leq C \left\{ \| (\Psi, Q) \|_{\dot{H}^{m+1} \times \dot{H}^{m+3}} + \| \Psi^2 \|_{\dot{H}^{m+3}} + \| Q \|_{\dot{H}^{m+2}}^2 \\
+ \| (\Psi, Q) \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}} + \| (\Psi, Q) \|_{\dot{H}^{m+4} \times \dot{H}^{m+4}} + \| \Psi^2 \|_{\dot{H}^{m+4}} + \| Q \|_{\dot{H}^{m+3}}^2 \\
+ \| (\Psi, Q) \|_{\dot{H}^{m+2} \times \dot{H}^{m+2}} + \| (\Psi, Q) \|_{\dot{H}^{m+6} \times \dot{H}^{m+5}} + \| \Psi^2 \|_{\dot{H}^{m+5}} + \| Q \|_{\dot{H}^{m+5}}^2 \right\}
\]
for $m \geq 1$. Using the isometry of $L(t)$ is on $H^{s+1} \times \dot{H}^s$ (see [19]), deduce that
\[
\| \mathcal{R}_\epsilon \|_{\dot{H}^{m+1} \times \dot{H}^m} \leq \| \mathcal{F}_{osc}^{(1)} \|_{\dot{H}^m} + \| \mathcal{F}_{osc}^{(2)} \|_{\dot{H}^{m+1}} + \| \Psi \|_{\dot{H}^{m+2}} \\
+ \| Q \|_{\dot{H}^{m+1}} \| \mathcal{F}_{osc}^{(1)} \|_{\dot{H}^{m+2}} + \epsilon \| \mathcal{F}_{osc}^{(2)} \|_{\dot{H}^{m+1}} \| \mathcal{F}_{osc}^{(1)} \|_{\dot{H}^{m+2}} + \| \nabla_W \mathcal{F}_{osc} (t/\epsilon, W(t)) \cdot \mathcal{F}_R (W) \|_{\dot{H}^{m+1} \times \dot{H}^m}
\]
(5.7)
for $m \geq 1$. We rewrite (5.7) as
\[
\| \mathcal{R}_\epsilon \|_{\dot{H}^{m+1} \times \dot{H}^m} \leq C \left\{ \| \mathcal{F}_{osc} \|_{\dot{H}^{m+3} \times \dot{H}^{m+2}} + \| \mathcal{F}_{osc} \|_{\dot{H}^m \times \dot{H}^{m+1}} \| W \|_{\dot{H}^{m+2} \times \dot{H}^{m+1}} + \epsilon \| \mathcal{F}_{osc} \|_{\dot{H}^{m+3} \times \dot{H}^{m+2}} + \| W \|_{\dot{H}^{m+4} \times \dot{H}^{m+3}} \right\}.
\]
(5.8)

By combining the above inequalities and Proposition (5.1), we deduce the $L^2_{loc} ([0, \infty); \dot{H}^{m+1} \times \dot{H}^m)$ bound for $\mathcal{R}_\epsilon$.
\[
\int_0^T \| \mathcal{R}_\epsilon \|_{\dot{H}^{m+1} \times \dot{H}^m} dt \leq C \left\{ \| (\varphi^{in}, g^{in}) \|_{\dot{H}^{m+5} \times \dot{H}^{m+4}} + \| (\varphi^{in}, g^{in}) \|_{\dot{H}^{m+5} \times \dot{H}^{m+4}}^4 \right\} \times (\| (\varphi^{in}, g^{in}) \|_{\dot{H}^{m+5} \times \dot{H}^{m+4}}^2 + 1) \\
+ \epsilon^2 \left\{ \| (\varphi^{in}, g^{in}) \|_{\dot{H}^{m+5} \times \dot{H}^{m+4}}^2 + \| (\varphi^{in}, g^{in}) \|_{\dot{H}^{m+5} \times \dot{H}^{m+4}}^4 \right\}.
\]
(5.9)

Thus we have proved the following proposition if $(\varphi^{in}, g^{in}) \in \dot{H}^{m+5} \times \dot{H}^{m+4}$.

(5.2) Proposition Let $m \geq 1$. If $(\varphi^{in}, g^{in}) \in \dot{H}^{m+5} \times \dot{H}^{m+4}$, then the solution of the renormalization group equation (2.4a, b) satisfies
\[
W = (\Psi, Q) \in C ([0, \infty); \dot{H}^{m+5} \times \dot{H}^{m+4}) \cap L^2_{loc} ([0, \infty); \dot{H}^{m+6} \times \dot{H}^{m+5}),
\]
and there exists a constant $C$ independent of $\epsilon$ such that
\[
\| \mathcal{R}_\epsilon \|_{L^2_{loc} ([0, \infty); \dot{H}^{m+1} \times \dot{H}^m)} \leq C.
\]
(5.9)

Let $w_\epsilon$ be the solution of the initial value problem (1.5) and $U_\epsilon$ the approximate solution (2.14) satisfying (2.41) then the difference $\omega = w_\epsilon - U_\epsilon$ satisfies
\[
\frac{d\omega}{dt} + \frac{1}{\epsilon} L \omega = \mathcal{F}(\omega) + Q(\omega, U_\epsilon) + Q(U_\epsilon, \omega) - \epsilon \mathcal{R},
\]
(5.10)
\[
\omega |_{t=0} = 0.
\]
where $\omega = (\varphi, g)^t$, $\mathcal{U}_\epsilon = (\tilde{\varphi}_\epsilon, \tilde{g}_\epsilon)^t$ and $\mathcal{R}_\epsilon = (F_\epsilon, G_\epsilon)^t$ is given by (2.16). Multiplying the first and second components of (5.10) by $\Delta \varphi$ and $\frac{1}{\epsilon} \log(1 + \epsilon g)$ respectively, then integrating over $\Omega$ we obtain

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon^2} \Phi(\epsilon g) dx + \int_{\Omega} |\Delta \varphi|^2 dx \\
\leq \left| \int_{\Omega} \epsilon \nabla F_\epsilon \cdot \nabla \varphi dx \right| + \left| \int_{\Omega} \epsilon G_\epsilon \log(1 + \epsilon g) dx \right| \\
+ \left| \int_{\Omega} -\nabla \tilde{g}_\epsilon \cdot \nabla \varphi \frac{\log(1 + \epsilon g)}{\epsilon} dx \right| + \left| \int_{\Omega} -\tilde{g}_\epsilon \Delta \varphi \frac{\log(1 + \epsilon g)}{\epsilon} dx \right| \\
+ \left| \int_{\Omega} -\nabla g \cdot \nabla \tilde{\varphi}_\epsilon \frac{\log(1 + \epsilon g)}{\epsilon} dx \right| + \left| \int_{\Omega} -\Delta \tilde{\varphi}_\epsilon g \frac{\log(1 + \epsilon g)}{\epsilon} dx \right|. 
\]

(5.11)

where $\Phi(r) \equiv (1 + r) \log(1 + r) - r$ is a strictly convex function defined over $r > -1$. The Legendre transform of $\Phi$ is given explicitly by

\[
\Phi^*(s) = s \Phi'(r) - \Phi(r) = e^s - s - 1
\]

which is defined for all $s \in \mathbb{R}$. Evidently we have the Young's inequality

\[
rs \leq \Phi(r) + \Phi^*(s).
\]

Assume $-1 < m \leq \epsilon g$ a.e. in $\Omega$, then applying the Young's inequality to the functions $f$ and $\log(1 + \epsilon g)$ we have for $\alpha > 0$

\[
\int_{\Omega} |f| \log(1 + \epsilon g) dx = \frac{\alpha}{\epsilon^2} \int_{\Omega} \frac{1}{\alpha} \epsilon^2 f \cdot \log(1 + \epsilon g) dx \\
\leq \frac{\alpha}{\epsilon^2} \int_{\Omega} \Phi(\epsilon^2 |f|) + \Phi^*(|\log(1 + \epsilon g)|) dx.
\]

(5.12)

We decompose the second integral of the RHS of (5.12) into two parts and using the fact that $r - \log(1 + r) \leq \Phi(r)$ for $r \geq 0$ to obtain

\[
\int_{\Omega} \Phi^*(|\log(1 + \epsilon g)|) dx = \int_{\{\epsilon g \geq 0\}} (1 + \epsilon g) - \log(1 + \epsilon g) - 1 dx \\
+ \int_{\{\epsilon g < 0\}} (1 + \epsilon g)^{-1} + \log(1 + \epsilon g) - 1 dx \\
= \int_{\{\epsilon g \geq 0\}} \epsilon g - \log(1 + \epsilon g) dx + \int_{\{\epsilon g < 0\}} \frac{\Phi(\epsilon g)}{1 + \epsilon g} dx \\
\leq \int_{\{\epsilon g \geq 0\}} \Phi(\epsilon g) dx + \int_{\{\epsilon g < 0\}} \frac{\Phi(\epsilon g)}{1 + \epsilon g} dx \leq C \int_{\Omega} \Phi(\epsilon g) dx,
\]

where $C = \max\{1, 1/((m + 1))$. Therefore (5.12) becomes

\[
\int_{\Omega} |f| \log(1 + \epsilon g) dx \leq \frac{\epsilon^2}{2\alpha} \int_{\Omega} |f|^2 dx + \frac{C \cdot \alpha}{\epsilon^2} \int_{\Omega} \Phi(\epsilon g) dx.
\]

(5.14)

Using the Sobolev inequality and (5.14) we obtain from (5.11)
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon^2} \Phi(\varepsilon g) dx + \left(1 - \frac{\|\tilde{g}_\varepsilon\|_{L^\infty([0,T];\dot{H}^2)}^2}{2\alpha} \right) \int_{\Omega} |\Delta \varphi|^2 dx \\
\leq \frac{c^2}{2} \left( \|F_\varepsilon\|^2_{\dot{H}^1} + \|G_\varepsilon\|^2_{L^2} \right) \\
+ \left[ \left(1 + \|\tilde{g}_\varepsilon\|_{\dot{H}^3}\right) \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 dx + \left(C \|\tilde{g}_\varepsilon\|_{\dot{H}^3} + \alpha C \|\tilde{g}_\varepsilon\|_{L^\infty([0,T];\dot{H}^2)} \right) \\
+ (2 + C) \|\varphi_\varepsilon\|_{\dot{H}^4} + C \right] \int_{\Omega} \frac{1}{\varepsilon^2} \Phi(\varepsilon g) dx \right],
\]

(5.15)

where $\alpha$ is chosen to satisfy $\alpha = \frac{1}{2} \left( \|\tilde{g}_\varepsilon\|_{L^\infty([0,T];\dot{H}^2)} + 1 \right)$. Since $\omega_{|t=0} = 0$ we deduce from Gronwall's inequality that

\[
\int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon^2} \Phi(\varepsilon g) dx \\
\leq \left( \frac{c^2}{2} \int_0^T \|\mathcal{R}_\varepsilon\|^2_{\dot{H}^1 \times L^2} dt \right) \exp \left[ \bar{c} \int_0^T \left(1 + \|\tilde{g}_\varepsilon\|^2_{\dot{H}^3} + \|\varphi_\varepsilon\|^2_{\dot{H}^4} \right) dt \right]
\]

(5.16)

where $\bar{c}$ depends on $T, m$ and the initial condition $\varphi_{in}, g_{in}$. Using (5.9) and the embedding $H^2 \times H^1 \hookrightarrow H^1 \times L^2$ we have the bound of the RHS of the first term of (5.16)

\[
\|\mathcal{R}_\varepsilon\|^2_{L^2_{loc}([0,\infty); \dot{H}^1 \times L^2)} \leq C.
\]

(5.17)

The second term is a bounded by \( \exp(\bar{c} T + \bar{c} \|\mathcal{U}_\varepsilon\|^2_{L^2([0,T]; \dot{H}^4 \times \dot{H}^3)}) \). Assuming $\mathcal{U}_\varepsilon \in L^2([0,T]; \dot{H}^4 \times \dot{H}^3)$ we obtain from (5.16) (5.17)

\[
\sup_{t \in [0,T]} \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon^2} \Phi(\varepsilon g) dx \leq C \varepsilon^2, \quad -1 < m \leq \varepsilon g.
\]

(5.18)

Moreover, if $\varepsilon g$ is of order $O(1)$, then it follows from the strictly convexity of $\Phi$ that

\[
\sup_{t \in [0,T]} \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 + |g|^2 dx \leq C \varepsilon^2,
\]

(5.19)

or equivalent

\[
\sup_{t \in [0,T]} \|w_\varepsilon - \mathcal{U}_\varepsilon\|^2_{\dot{H}^1 \times L^2} \leq C \varepsilon.
\]

(5.20)

Therefore we obtain the following comparison theorem.

(5.3) **Theorem** Let $\Omega = (0,1)^N, N < 4$ be the periodic domain and $w_\varepsilon = (\varphi_\varepsilon, g_\varepsilon)$ the solution of (1.5) with initial data $(\varphi_{in}^\varepsilon, \tilde{g}_{in}) \in \dot{H}^6(\Omega) \times \dot{H}^5(\Omega)$. Then there exist constants $C, T > 0$ depending on initial data such that the function $\mathcal{U}_\varepsilon$ defined by (2.14) satisfying

\[
\mathcal{U}_\varepsilon \in C([0,\infty); \dot{H}^3 \times \dot{H}^2) \cap L^2_{loc}([0,\infty); \dot{H}^4 \times \dot{H}^3)
\]

(5.21)

and the difference $(\varphi, g)$ between $w_\varepsilon$ and $\mathcal{U}_\varepsilon$ satisfying
\[
\sup_{t \in [0,T]} \int_{\Omega} \frac{1}{2} |\nabla \psi|^2 + \frac{1}{\varepsilon^2} \Phi(\varepsilon \rho) \, dx \leq C \varepsilon^2
\]
providing \(-1 < \nu \leq \varepsilon \rho\) a.e. in \(\Omega\). In addition, if \(\varepsilon \rho\) is of order \(O(1)\), then
\[
\sup_{t \in [0,T]} \|w_t - \bar{U}_t\|_{H^1 \times L^2} \leq C \varepsilon.
\]

We can also justify the formal derivation of (2.45) and (2.46) for the well-prepared initial data.

(5.4) Corollary For the case of well-prepared initial data, i.e. \(\Delta \rho^0 = 0\), \(\rho^0 = 0\), then density fluctuation is identically zero, \(Q = 0\) and the potential \(\Psi\) satisfies the Bernoulli's equation,

\[
\frac{d\Psi}{dt} + \Pi = 0
\]
where

\[
\Pi = \sum_{k \in \mathbb{Z}^N} e^{ik \cdot x} \left( \frac{1}{8} \sum_{\alpha, \beta, \gamma = \pm 1} \sum_{\sigma \mid j + \sigma = k, |j| + |\sigma| + |\gamma| = 0} (\nabla \Psi)_j \cdot (\nabla \Psi)_l \right).
\]

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