DEVELOPMENT AND GEOMETRIC SIMILARITY OF ALLUVIAL DELTAS

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ABSTRACT: Movable bed experiments from a flume into a basin were conducted to study the development of alluvial deltas. The experimental setup was aimed at the bed-load mode. Results showed that the development of the delta can be divided into three stages. In the first stage, the delta progressed mainly in its length, and a successfully derived equation described the shape of the delta in this stage. In the second stage, the delta developed mainly in its width. The length, width, front thickness, and central thickness of the delta were used to scale the geometric similarity. The plane geometry of deltas can be described using Gaussian functions. Transverse and longitudinal profiles fit the hyperbolic and linear function, respectively. The length, width, and thickness of the delta deposition at the end of the second stage have their own maximum values, \( R_b, B_c, \) and \( Z_c \), respectively, for each case. The total volume \( V \) of sediment deposition could be simply related to these maximum values by \( V = \alpha R_b B_c Z_c + \beta \) with shape parameters \( \alpha \) and \( \beta \) around 0.178 and 174. The third stage was characterized by the interactions between the delta development and stream-channel variations. The ratios of the delta slope to the channel slope were closely related to the occurrence condition of the stream channel. Finally, the temporal variations of stream channels were investigated qualitatively.

INTRODUCTION

Alluvial deltas in estuaries are one of the most heavily utilized regions for industrial, commercial, recreational, and municipal activities. There have been numerous studies on river delta problems. Most of these studies were based on field investigations, laboratory experiments, and numerical simulations. For example, Wright and Coleman (1974) classified river deltas into bar back, bar crest, bar front, and distal bar according to field observations from the Mississippi River. Chang (1982) used experiments and the theory of minimum stream power to study the stream-channel hydraulics on a river delta, but the condition of stream-channel occurrence on a delta was not discussed. Tanaka et al. (1987) used sine and cosine functions to describe the plane shape of a river delta. His writing, however, did not clearly specify under which developing stage of the delta these results were obtained. Ashida et al. (1987, 1988, 1989) used a series of experiments to study the formation processes of a river delta and the variations of the stream channels on top of the delta. In their studies, however, the geometric similarity of the delta was not clearly stated. To investigate these problems, the deltas in this study were simplified as a topography process caused only by sediment deposition in a wide basin, whereas the influences of waves, tides, and density differences were excluded. The temporal variation of a shape factor was employed to divide the development of the delta into three stages. In the first stage, the velocity profiles of a plane jet and the critical shear stresses for the incipient motion of sediment were used to describe the configuration of the deltas. In the second stage, the length, width, front thickness, and central thickness of the delta were adopted as characteristic scales to analyze the similarity of alluvial deltas. Finally, the occurrence and variation of the stream channels upon the delta were investigated in the third stage, which could provide a prospect in understanding the river migration processes on the Yellow River delta.

EXPERIMENTAL PROCEDURES

Experimental Setup

The experimental setup, shown in Fig. 1, consisted of a wide basin and a rectangular channel. Water and sediment supply systems were installed at the upstream end of the channel. An adjustable weir was constructed at the downstream end of the basin for controlling the water levels. To measure the topography of the delta and the water depth, the water level and bed level gauges were set on a carriage, which was controlled by a host computer, and could move in the longitudinal and lateral directions. For observing the flow field, paper disks, used as tracers, were released onto the water surface. The trajectories of these tracers were recorded by a video camera, and then transferred to a flow velocity field using an image processor.

Experimental Conditions

Table 1 lists the experimental conditions of 14 runs. In this table, \( Q_w \) and \( Q_s \) are the water discharge and the rate of sediment feed, respectively. The parameters \( S_o \) and \( S_{bi} \) are the initial longitudinal bed slopes of the channel and basin, respectively. Note that \( S_o \) is always larger or equal to \( S_{bi} \) in our setting. The parameters \( L_c \) and \( W_c \) are the length and width of the channel; \( L_b \) and \( W_b \) are the length and width of the basin, respectively; and \( d \) is the diameter of the sediment feed. In each run, natural sand, with uniform size, was used as the sediment material and the movable bed material. The water and sediment were supplied at a constant rate from the upstream end of the channel. The flow conditions in the flume were controlled to be as close to longitudinally uniform as possible. The transport of sediment was confined in the bed-load mode by controlling the parameters \( u_s/u_{so} \geq 1.0 \) and \( u_{so}/\alpha_o < 1.0 \), in which \( u_s \) is the shear velocity in the channel; \( u_{so} \) is the incipient shear velocity of the sediment particles; and \( \alpha_o \) is the fall velocity of sediment particles. To achieve these two criteria, the flow discharge and the sediment-feed rate were first determined according to the equipment capacities, respectively. The shear velocity \( u_s \) and the initial channel slope \( S_o \) were decided by solving the bed-load formula and Manning’s formula simultaneously. During the experiments, the channel depth as well as the topography of the channel bed were measured and double-checked with the calculated topography. In each case, the water level in the downstream end of the basin was also controlled at a constant level, which was set as the normal depth elevation of the flume outlet.
FIG. 1. Layout of Experimental Equipment

TABLE 1. Experimental Conditions

<table>
<thead>
<tr>
<th>Run number</th>
<th>( Q_w ) (cm(^3)/s)</th>
<th>( Q_s ) (g/s)</th>
<th>( d ) (mm)</th>
<th>( S_c ) (g/mm)</th>
<th>( S_w ) (g/mm)</th>
<th>( L_c ) (cm)</th>
<th>( W_c ) (cm)</th>
<th>( L_b ) (cm)</th>
<th>( W_b ) (cm)</th>
<th>( T_p ) (min)</th>
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<td>1.00</td>
<td>0.7</td>
<td>0.003</td>
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<td>30</td>
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<td>0.001</td>
<td>150</td>
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Note: NA = not available.

PROCESSES OF DELTA FORMATION AND DISCUSSION

Development of Delta

The results of Run A-10 were used as an example to explain the development of deltas. Figs. 2(a and b), 3, and 4 illustrate the surface velocity, plane shape, longitudinal and transverse profiles of the delta at times of 0, 30, 80, and 370, min, respectively.

As shown in Fig. 2(b), the delta developed mainly in its length, and the delta shape was similar to the shape of a tongue at \( t = 30 \) min. At \( t = 80 \) min, the delta became wider, but its length was almost the same as that at \( t = 30 \) min. The delta shapes were symmetrical both at \( t = 30 \) and \( t = 80 \) min. At \( t = 370 \) min, a stream channel appeared, and the delta resulted in a nonsymmetrical shape.

As shown in Fig. 2(a) the velocity profiles at \( t = 0 \) min were similar to those of a jet flow with bottom friction. After the delta developed, the flow boundary became wider, and the velocity along the centerline decreased at \( t = 30 \) min. Further-
FIG. 2. (a) Transverse Velocity Profiles of Surface Flows for Case of Run A-10; (b) Plane Shape of Delta for Case of Run A-10
more, at $t = 80$ min, the velocity profile became a double-peak distribution in front of the delta, which indicates the blocking effect of the delta front. At $t = 370$ min, the velocity profiles were obviously affected by the stream channel, and the maximum velocities of the flow shifted to the stream channel.

Fig. 3 shows the bed profiles along the centerline. In this figure, the origin of the $x$-coordinate ($x = 0$) is the position of the channel outlet. It was found that the slope of the delta surface gradually became reversed with time, but the channel bed still kept its initial slope.

As shown in Fig. 4, the shapes of the transverse profiles were similar to a trapezoidal platform from $t = 30$ min to $t = 80$ min. But after the stream channel appeared ($t = 370$ min), the shapes became nonsymmetrical.

Fig. 5 shows the temporal variations of the length of the center line $R_c$, the maximum with $B_m$, and the front elevation $Z_f$ of the delta. In Fig. 5, time lags exist between the evolution for different dimensions, and this can be noted by the sections of flat slopes and steep slopes of each curve located within different time periods.

Since the stream channel greatly affected the growing deltas, the ratio of the temporal variations $S_m/S_c$ were used in this study to describe the variation of the stream channel as shown in Fig. 6, where $S_m$ is defined as the maximum slope on the delta surface along the centerline, and $S_c$ is the average slope of the channel bed. According to the experimental results, the local minimum of $S_m/S_c$ ($A$, $C$, $E$, and $G$) were the points where the stream channel occurred. $AB$, $CD$, and $EF$ represented the migration process of the stream channel, while the local maximum of $S_m/S_c$ ($B$, $D$, and $F$) were the points where the migration of the stream channel stopped, and $BC$, $DE$, and $FG$ indicated the process in which the sediment deposited in the stream channel. At the end of each period of $BC$, $DE$, and $FG$ (i.e., the points of $C$, $E$, and $G$), new stream channels occurred. These stream channels repeated the cycle of migration-stop-deposition-occurrence as long as the experiment continued.
FIG. 6. Temporal Variations of $S_m/S_c$ for Case of Run A-10

FIG. 7. Temporal Variations of Shape Factor

Stages in Development of Deltas

To study the stages in delta development, the shape factor $R_c/B_m$ and the time factor $t/T_p$ were employed and shown in Fig. 7. In this figure, $T_p$, listed in Table 1, is the time when the shape factor reaches its peak. The first stage is defined as the period when the shape factor ranged from zero to a peak, and the second stage is defined as the period when the shape factor decreased from a peak to unity. After the second stage, the rest of the growing delta is defined as the third stage. In the third stage, which was affected by channelization, the shape factor oscillated, and the time average of the oscillation was near 1.0, which implied that the averaged delta length and width were nearly the same during the oscillating period.

THEORETICAL ANALYSIS FOR DELTA MORPHOLOGY IN FIRST STAGE

To study the morphology of an alluvial delta in the first stage, the coordinate system shown in Fig. 8 was used. As mentioned above, the surface velocity of the centerline and transverse profiles in the first stage were similar to those of the jet flow shown in Figs. 9 and 10. The classical plane jet theory with a uniform depth and frictionless bottom can be expressed as (1) and (2), respectively

$$U_c(x) = C_1U_o \sqrt{\frac{B_o}{x}}, \quad x \geq L_p$$  \hspace{1cm} (1)

$$\frac{U_c(x,y)}{U_c(x)} = \exp \left( -\frac{y^2}{2C_2x^2} \right), \quad x \geq L_p$$  \hspace{1cm} (2)

where $U_c$ = centerline velocity; $U_c$ = velocity at position $(x, y)$; $U_o$ = velocity at the channel outlet; $L_p$ = length of potential core, in which the velocity did not decay and was almost kept as $U_c$; $C_1$ and $C_2$ = constants; and $B_o$ = half-width of the channel. Under consideration of bottom friction and variable bathymetry, Joshi (1982) derived a theoretical flow field distribution of a tidal jet. The tidal jet is treated as a vertically averaged flow, and the effects of lateral mixing, bottom fric-
The centerline and transverse velocity profiles for the linearly sloping bottom in the fully developed region can be expressed as (3) and (4), respectively:

\[
U_c(x) = \left( \frac{D}{C} \right)^{1/2} h_o^{1/6} \left( \frac{2s - F}{8} \right)^{1/2} \left[ (h_o + sx)^{2 - F/8s} - h_o^{2 - F/8s} \right]^{1/2} 
\]

\[
U_c(x, y) = f \left( \frac{y}{b(x) \sqrt{2}} \right) 
\]

where \( b(x) = \) half-width of jet; \( D = \) axial jet momentum flux at the inlet equal to \( h_o B_o U_{c,i}^2 \); \( h_o = \) water depth at channel outlet; \( C = \) constants; \( F = \) Darcy-Weisbach friction factor; and \( s = \) bottom slope of basin. It was found that the bottom friction causes the centerline velocity to decay more rapidly and the jet to expand faster laterally. The velocities defined for both plane jet and tidal jet from (1) to (4) are all depth-averaged velocities. In the present study, for simplicity, all of the velocities mentioned in (1)–(3) and (4a) were treated as surface velocities. To compare the applicability of these two theories, measured data are plotted against the theories in Fig. 11, because the plane jet theory appears more suitable than the tidal jet theory to describe the flow field in the present study. Although the plane jet theory was obtained under the conditions of frictionless bottom and uniform depth, the effect of bottom friction and varying bathymetry can be included by adjusting the distribution coefficients. The experimental results by Albertson et al. (1950) indicated that \( C_1 = 3.22, C_2 = 0.109, \) and \( L_p = 10B_o \). In this study, obtained values are \( C_1 = 2.7, C_2 = 0.12, \) and \( L_p \) ranged from \( 6B_o \) to \( 8B_o \).

Furthermore, the velocity profile in depth was assumed to satisfy the logarithmic profile for a rough wall, and the surface velocity therefore could be written:

\[
\frac{U_c}{U_s} = \frac{1}{\kappa} \ln \left( \frac{h_o + sx}{k_s} \right) = A, 
\]

where \( U_s = \sqrt{\tau_o / \rho} = \) shear velocity; \( \rho = \) water density; \( \tau_o = \) bottom shear stress; \( \kappa = \) von Kármán constant (=0.4); \( h_o = \) water depth at the channel outlet; \( k_s = \) equivalent sand roughness \( (k_s = d \text{ was assumed}); \) and \( A = \) coefficient (=8.5). Sub-

![FIG. 8. Definition Sketch of Velocity Profiles in First Stage](image)

![FIG. 9. Centerline Velocity Distribution in First Stage](image)

![FIG. 10. Transverse Velocity Distribution in First Stage](image)
Fig. 11. Comparison of Centerline Velocity between Plane Jet Theory and Tidal Jet Theory

Fig. 12. Comparison of Delta Plane Shape for Case of Run A-10

By substituting (1) and (2) into (5), the bottom shear stress in the $x$-direction was derived as

$$\tau_x = \frac{C_d}{x} \exp \left( -\frac{y^2}{2C_d^2x} \right)^2, \quad x \approx L_p$$  \hspace{1cm} (6a)

in which

$$C_d = \frac{\rho L_p U_0^2 C_1^2}{\left( \frac{1}{\kappa} \ln \frac{h_s + sx}{k_c} + A_c \right)^2}$$ \hspace{1cm} (6b)

In this study, $\tau_x = \tau_c$ was assumed as the stop criteria of the bed-load support, where $\tau_c$ is the critical shear stress for the incipient motion of the bed load. In the first stage of evolution, the flow type is similar to the jet flow, and the dominating components are in the $x$-direction. The $y$-component is relatively negligible. Therefore, the plane shape of the delta was able to predict using the shape of the shear diagram for $\tau_x = \tau_c$. An empirical formula for the incipient motion of bed load suggested by Iwagaki (1956) is as follows:

$$\frac{\tau_c}{\rho} = 55d, \quad \text{0.565 mm} \leq d \leq 1.18 \text{ mm}$$  \hspace{1cm} (7)

By substituting (7) into (6a), the plane shape of the delta for the end of the first stage can be described as follows:

$$\ln x + \left( \frac{y}{C_d^{2}x} \right)^2 = \ln A_c, \quad x \approx L_p$$  \hspace{1cm} (8a)

Fig. 12 demonstrates similar trends of (8) and the experimental results of Run A-10. Fig. 13 shows a good agreement of delta length between theoretical and experimental results.

GEOMETRIC SIMILARITY FOR DELTA MORPHOLOGY IN SECOND STAGE

To analyze the similarity of deltas in the second stage, the plane shape of the delta was first described by the polar coordinates $R$ and $\theta$, as shown in Fig. 14, where $R = \text{distance from the channel outlet to the edge of the delta}$; and $\theta = \text{expansion angle from the centerline}$. The plane shape of the deltas were then analyzed and plotted in Fig. 15 using the parameters of $\theta = \theta_s$ and $R = R_s/R_c - R_c$, where $\theta_s = \text{expansion angle measured from the origin to the maximum width of the delta}$; and $R_s = \text{value of } R \text{ at } \theta = \pm 90^\circ$. In Fig. 15, the Gaussian function was found to have a good agreement with the experimental results

$$\frac{R - R_c}{R_c - R_s} = \exp \left( -\frac{\theta^2}{2C_g^2\theta_s^2} \right)$$  \hspace{1cm} (9)

where $C_g =$ coefficient. In the case of Fig. 15, $C_g = 0.55$ is obtained. Comparison between the Gauss distribution formula to the dimensionless form of the delta shapes [(9)] indicates that the coefficient $C_g$ can represent the standard deviation $\sigma$ of the dimensionless form of the delta shapes ($C_g \approx \sigma$ in the exponential term; $1/\sigma \sqrt{2\pi} = 1.0$ and $1/C_g \sqrt{2\pi} = 0.73 \approx 1.0$). Several subsequent experiments indicated that the value of $C_g$ can be affected by the sediment gradation used in experiment.

The process of determining the $C_g$ value is as follows. Let $P(X_i, Y_i)$ be an arbitrary point of the experimental results in Fig. 15, and let $P(X_i, \hat{Y}_i)$ be a point on the Gaussian curve, which had the same $X_i$ coordinate with the point $P$. Then the distance between $P$ and $P'$ was determined using

$$|\ln Y_i - \ln \hat{Y}_i| = \ln \left| Y_i + \frac{X_i^2}{2C_g} \right|$$  \hspace{1cm} (10)

Hence the sum of the squares of the distances of all experimental points was

$$E = \sum_{i=1}^{n} \left[ (\ln Y_i)^2 + \frac{1}{C_g^2} X_i^2 \ln Y_i + \frac{1}{4C_g^2} X_i^2 \right]$$  \hspace{1cm} (11)

To obtain the optimal value of $C_g$ for the Gaussian function, the distance between $P$ and $P'$ should be minimum. By letting the derivative of $E$ with respect to $C_g$ be zero, the distance
between $P$ and $P'$ was minimized, and the optimal $C_e$ was then obtained as follows:

$$C_e = \sqrt{-\sum_{i=1}^{\infty} \frac{X_i^t}{2 \sum_{i=1}^{\infty} X_i^t \ln Y_i}} \quad (12)$$

The coefficient of correlation between the Gaussian curve and the experimental points was defined as

$$\gamma^2 = 1 - \frac{E}{\sum_{i=1}^{\infty} (\ln Y_i)^2} \quad (13)$$

Substituting the experimental data into (12) and (13), the optimal value of $C_e$ and the coefficient of correlation $\gamma$ were obtained as 0.55 and 0.9, respectively.

Furthermore, to analyze the similarity of the transverse shape of the delta, the coordinates and parameters are defined as Fig. 16, where $y = 0$ represents the center point of the transverse, and the positive direction of the $y$-axis is to the right; $B$ = width of the transverse; and $Z$ and $Z_c$ = thickness of the transverse at the coordinate of $y$ and $y = 0$, respectively. By adopting the parameters of $Z/Z_c$ and $y/B$, the transverse shape of the deltas were then plotted as shown in Fig. 17. A hyperbolic function as in (14) was found to be appropriate to fit the transverse of the deltas

$$2 \left( \frac{Z}{Z_c} - \frac{1}{2} \right) = \tanh \left( \frac{y}{B} \right) + S_1 \quad (14)$$

where $S_1$ and $S_2$ = coefficients to be determined.

By the same procedure for obtaining the optimal value of $C_e$, the optimal value of $S_1$ and $S_2$ were obtained as $-11.0$ and $5.0$, respectively. The coefficient of correlation $\gamma$ between the theoretical and experimental results was 0.91.

To analyze the similarity of longitudinal shape of the delta, the coordinates and parameters are defined as Fig. 18, where $Z_c =$ deposition depth at any $x$ in the centerline direction; and $Z_f =$ maximum deposition depth in the centerline direction. Using the parameters of $Z/Z_c$ and $x/R_c$, the nondimensional longitudinal shape of the deltas could be plotted in Fig. 19. A similar linear function was found to be appropriate to describe the longitudinal shape of the deltas as follows:

$$Z(x, R_c) = L_1 \left( \frac{x}{R_c} \right)_0 \quad (15a)$$

$$Z(x, R_c) = L_1 \left( \frac{x}{R_c} \right) + L_2, \quad 0.95 \leq \frac{x}{R_c} \leq 1.0 \quad (15b)$$

where $L_1$, $L_2$, and $L_3$ = coefficients, which could be obtained following the same procedure mentioned above, as $1.1$, $-21.0$, and $21.0$, respectively.

Substituting (15) into (14) yields the 3D deposition depth $Z(x, y)$ as follows:
In (9), \( R \) denoted the outline of the delta; therefore the delta width \( B \) may be expressed as

\[
B = 2R \sin \theta = 2 \sin \left( R + (R_s - R_c) \exp \left( -\frac{\theta^2}{2C_s\theta_s} \right) \right)
\]

(17)

\[
Z(x, y) = \frac{1}{2} Z_c \left[ L_z \left( \frac{x}{R_c} \right) + L_s \right] \left[ 1 + \tanh \left( S_1 \frac{y}{B} + S_2 \right) \right],
\]

(16b)

\[
0 \leq \frac{x}{R_c} \leq 0.95
\]

(16a)
where expansion angle $\theta_a$ could be related to the factor $B_m/R_c$ [i.e., $\theta_a = f(B_m/R_c)$]. In this paper, the following relation could be found from the experimental results:

$$\theta_a = 62 \left( \frac{B_m}{R_c} \right) + 12, \quad 0.5 \leq \frac{B_m}{R_c} \leq 0.75$$

Substituting (17) into (16) and translating the Cartesian coordinates to polar coordinates yields

$$Z(r, \theta) = \frac{1}{2} Z_f \left( \frac{L_r r \cos \theta}{R_c} \right) + \frac{1}{2} \left[ 1 + \tanh \left( \frac{S_t r}{2R_c + 2(R_c - R_e) \exp \left( \frac{-\theta^2}{2C_g^2 S_2} \right)} \right) \right].$$

$$0 \leq \frac{x}{R_c} \leq 0.95$$

The volume of an alluvial delta in the present study can be determined by

$$V = 2 \int_0^{\pi/2} \int_0^{0.95R} Z(r, \theta) r \, dr \, d\theta \approx 2 \int_0^{\pi/2} \int_0^{0.95R} Z(r, \theta) r \, dr \, d\theta$$

Note that the volume between 0.95$R$ and 1.0$R$ is neglected. It is still difficult to give an exact solution of (20) due to the nonlinearity of the function. The Simpson's numerical integral method was used to evaluate (20), and the volume of the alluvial delta for the end of the second stage could be approximated by

$$V = \alpha R_e B_m Z_f + \beta$$

with $L_1 = 1.1$, $C_g = 0.55$, $S_1 = -11.0$, and $S_2 = 5.0$, the values of $\alpha$ were between 0.160 and 0.195 and can be treated as 0.178. Coefficient $\beta$ in (21) is the residual term in the process of integration of (20). In this study, $\beta$ is between 100 and 300 and is not a constant. The value of $\beta$ was taken as 174 in this study for prediction purposes. A comparison of the measured total volume of alluvial deltas and the calculated one by using (21) is indicated in Fig. 20. By comparison, one can see that (21) can give a good estimate of the total volume of an alluvial delta.
OCCURRENCE AND VARIATION OF STREAM CHANNELS IN THIRD STAGE

The existence of channel flow plays an important role in the whole formation processes of alluvial deltas. Parker et al. (1998) indicated that channelization acts to lower the slope at each point on the fan below the value that would prevail for unchannelized sheet flow. A similar phenomenon was also found in the alluvial delta system in the present study. According to the results of Run A-10, the occurrence of the stream channel was always accompanied with the local minimal value of $S_m/S_c$. This result was also found in the other runs, although no theoretical explanation was found yet. To understand the occurrence condition of the stream channel, the $S_m/S_c$ at the first appearance of the stream channel is shown in Table 2. According to this table, the first stream channel occurred when $S_m/S_c$ approached the range between $-2$ and $-3$ in the experiments here. The value of $S_m/S_c$ at the following occurrence of the stream channel was plotted in Fig. 21. According to this figure, the negative slope of $S_m/S_c$ is the steepest the first time that channelization occurs. At that moment, the flow system shifted from sheet flow to channel flow regime. The absolute value of slope ratio $S_m/S_c$ decreased for the following occurrence of channelization.

Initially, the stream channels on a delta always find a shortcut through either the right- or left-hand side of the centerline. After it occurred, however, the stream channel tended to migrate toward the centerline of the delta. In Fig. 22, the outer bank of the stream channel moved from the side of the delta to the centerline during a period of 183 min. At $t = 183$ min, the stream channel became nearly a straight line, and such a migration process was defined as the “straightening process” of the stream channel in this study. The outer bank erosion and inner bank deposition were clearly found; these two phenomena allowed the stream channel to keep its straightening process continuously. The straightening process of the stream channel can provide a prospect for the Yellow River delta from 1904 to 1986 as shown in Fig. 23 (Long and Qian, 1986). A stream channel took its course in position A in 1904 and flowed in that course continuously until 1917. From 1917 to 1929, the stream channel changed its position to the centerline of the delta. After 1929, the river stream took its course in position D and then changed from position D to the centerline of the delta (position G). Although the dynamic environments are extremely different for Yellow River delta compared to the experiments here, the formation mechanism is similar in terms of the developing stages. For example, the major difference is that many channels on the Yellow River delta are artificially adjusted. Since that area is the second largest oil field in mainland China, a large amount of protection works and artificial channelization restricted the natural evolution. The other main differences are the discharge variation of Yellow River and the downstream conditions of tide, ebb, and alongshore currents. Although considerable artificial effects as well as downstream current effects provide different environments on the Yellow River delta, the straightening tendency of the channel did exist, as evidenced in historical records.

**TABLE 2.** $S_m/S_c$ Values for First Stream Channel Occurrence

<table>
<thead>
<tr>
<th>Run number</th>
<th>$S_m/S_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>A-2</td>
<td>-2.0</td>
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<td>A-8</td>
<td>-2.2</td>
</tr>
<tr>
<td>A-10</td>
<td>-2.8</td>
</tr>
</tbody>
</table>

**FIG. 20.** Comparison of Measured and Calculated Total Volumes of Deltas

**FIG. 21.** Value of Slope Factor at Following Occurrence of Stream Channel
SUMMARY AND CONCLUSIONS

Based on a series of flume experiments, mainly with bedload transport and without the influence of waves, tides, and density difference, the development of an alluvial delta can be divided into three stages using a criterion of temporal variations of the shape factor $R_c/B_m$. The first stage can be defined as the longitudinal growing period, in which the delta grows rapidly in the longitudinal dimension. The shape factor ranged from zero to a peak in this period. During this stage, the delta advanced mainly in its length, and the plane shape of its limit can be predicted by (8a) based on the concept of the incipient motion of particles.
The second stage can be defined as the period when the shape factor decreased from the peak back to unity. In this stage, the delta mainly progressed in its width, and its plane geometry, transverse, and longitudinal profiles agreed very well with the Gaussian, hyperbolic, and linear functions, respectively. Eq. (21) is derived to quantify the total amount of deposition simply by the dimensions of the delta.

Following the second stage, the remainder of the delta’s development was known as the third stage. In the third stage, channelization occurs on the top of a well-grown delta. As affected by the stream channels, the shape factor oscillated with an average time near 1.0 and resulted in a nonsymmetrical shape. The first stream channel occurred when the slope ratio $S_0/S_1$ approached a minimum. The stream channel, after it occurred, tended to migrate toward the centerline of the delta accompanied by processes such as side erosion and retrogressive deposition and was named the straightening process. After the straightening process, another channel emerged and repeated a similar cycle. The third stage of delta development is important because most natural deltas in the world are in that stage of development.

APPENDIX I. REFERENCES


APPENDIX II. NOTATION

The following symbols are used in this paper:

$A_0 = \text{constant in rough logarithmic velocity profiles}$; 
$B, B_w = \text{width and maximum width of delta}$; 
$B_c = \text{half-width of channel}$; 
$b(x) = \text{half-width of jets}$; 
$C = \text{coefficient relating to velocity distribution of tidal jets}$; 
$C_e = \text{coefficient relating to Gaussian curves for similarity analysis}$; 
$C_1, C_2 = \text{longitudinal and transverse distribution coefficients of surface velocity of plane jets, respectively}$; 
$D = \text{axial jet momentum at channel outlet}$; 
$d = \text{diameter of sediment}$; 
$F = \text{Darcy-Weisbach friction factor}$; 
$h = \text{water depth}$; 
$h_e = \text{water depth at channel outlet}$; 
$k_e = \text{equivalent sediment roughness}$; 
$L_b, L_c = \text{length of basin and channel, respectively}$; 
$L_p, L_{cp} = \text{coefficients relating to simple linear function for similarity analysis}$; 
$Q_w, Q = \text{upstream water discharge and sediment feeding rate of channel}$; 
$R, \theta_0 = \text{outline of deltas in polar coordinates}$; 
$R_0 = \text{delta length}$; 
$R_{cp}, R_e = \text{experimental result and theoretical prediction of delta length in first stage}$; 
$R_t = \text{value of } R \text{ at } \theta = \pm 90^\circ$; 
$S_e = \text{average slope of channel}$; 
$S_0, S_1 = \text{initial longitudinal bed slope of channel and basin, respectively}$; 
$S_c, S_e = \text{coefficients relating to hyperbolic function for similarity analysis}$; 
$s = \text{bottom slope of basin}$; 
$T_e = \text{time when shape factor reaches maximum value}$; 
$U_0 = \text{surface velocity along centerline in first stage}$; 
$U_x = \text{surface velocity at position } y \text{ in first stage}$; 
$U_y = \text{surface velocity at outlet of channel in first stage}$; 
$u_x = \text{shear velocity of water}$; 
$u_{se} = \text{incipient shear velocity of sediment particle}$; 
$W, W_0 = \text{width of basin and channel, respectively}$; 
$x = \text{longitudinal coordinate}$; 
$y = \text{transverse coordinate}$; 
$z = \text{vertical coordinate}$; 
$\alpha, \beta = \text{coefficients relating to deposition volume of deltas}$; 
$\gamma = \text{coefficient of correlation}$; 
$\theta_e = \text{expansion angle measured from origin to maximum width of delta } B_e$; 
$\kappa = \text{von Kármán constant}$; 
$\rho = \text{water density}$; and 
$\tau_s = \text{bottom shear stress}$.