Extensions of Neveu–Schwarz conformal modules
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In this paper we study extensions between two finite-rank irreducible conformal
modules over the Neveu–Schwarz algebra. © 2000 American Institute of Physics.

I. INTRODUCTION

Finite-rank conformal modules over the Virasoro, the current, and the Neveu–Schwarz alge-
bras were studied in Ref. 1. In particular, finite-rank irreducible conformal modules over these
(super)algebras were classified using the equivalent languages of conformal superalgebras and
extended annihilation algebras. However, conformal modules are in general not completely reduc-
ible, and thus one needs to consider the extension problem. In Ref. 2 we classified extensions
between two finite-rank irreducible conformal modules over the Virasoro algebra. The purpose of
the present paper is to do the same for the Neveu–Schwarz algebra. The method here, as in Ref.
2, is based on the theory of conformal superalgebras.1,3 Since the Virasoro algebra is the even part
of the Neveu–Schwarz algebra, we use heavily the classification of extensions of Virasoro con-
formal modules obtained in Ref. 2.

The organization of this paper is as follows. We first review the notion of a formal distribution
Lie superalgebra and that of a conformal superalgebra and describe their connection with each
other.7 After that we describe briefly the aspects of representation theory of the Virasoro and the
Neveu–Schwarz algebras (and the associated conformal superalgebras) that will be important for
the rest of the paper. In particular the two main tools, namely the classification of the irreducible
conformal modules over the Neveu–Schwarz (conformal) algebra1 and the classification of exten-
sions of conformal modules over the Virasoro (conformal) algebra,2 will be recalled. Section III is
then devoted to classification of extensions of Neveu–Schwarz conformal modules in the case
when one of the modules involved in the extension is one-dimensional. In the last section the case
when both modules involved in the extension are non-one-dimensional is considered. Throughout
our discussion all vector spaces, tensor products, and algebras are assumed to be over the field of
complex numbers C.

II. PRELIMINARIES

A. Formal distributions and conformal superalgebras

A formal distribution with coefficients in a vector space U is a generating series of the form

\[ a(z) = \sum_{n \in \mathbb{Z}} a_{[n]} z^{-n-1}, \]

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Two formal distributions \(a(z)\) and \(b(z)\) with coefficients in a Lie superalgebra \(g\) are called (mutually) local if for some \(N \in \mathbb{Z}_+\) one has
\[
(z-w)^N [a(z), b(w)] = 0.
\] (2.1)

Consider the formal delta function
\[
\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n.
\]

Then condition (2.1) is equivalent to the operator product expansion:
\[
[a(z), b(w)] = \sum_{j=0}^{N} (a_{(j)} b)(w) \frac{\partial^j}{\partial w^j} \delta(z-w)
\] (2.2)

[here \(\partial_w^j\) stands for \((1/j!) \partial^j/\partial w^j\)]

for some formal distributions \((a_{(j)} b)(w)\) (Ref. 3, Theorem 2.3), which are determined by the formula
\[
(a_{(j)} b)(w) = \text{Res}_z (z-w)^j [a(z), b(w)].
\] (2.3)

Formula (2.3) defines a \(\mathbb{C}\)-bilinear product \(a_{(j)} b\) for each \(j \in \mathbb{Z}_+\) on the space of all formal distributions with coefficients in \(g\).

The space of all formal distributions with coefficients in \(g\) is naturally a (left) module over \(\mathbb{C} [\partial_z]\). We have \(\partial_z a(z) = \Sigma_n (\partial a)(_n) z^{-n-1}\), where \((\partial a)(_n) = - na([n-1])\).

A Lie superalgebra \(g\) is called a formal distribution Lie superalgebra, if there exists a family \(\mathfrak{g}\) of pairwise local formal distributions whose coefficients span \(g\). We will write \((g, \mathfrak{g})\) for such a Lie superalgebra.

A well-known example is the (centerless) Virasoro algebra, the Lie algebra \(\mathfrak{V}\) with basis \(L_n (n \in \mathbb{Z})\) and commutation relations
\[
[L_m, L_n] = (m-n) L_{m+n}.
\]
It is spanned by the coefficients of the local formal distribution \(L(z) = \Sigma_n \mathbb{Z} L_n z^{-n-2}\) satisfying
\[
[L(z), L(w)] = \partial_z L(w) \delta(z-w) + 2 L(w) \partial_z \delta(z-w).
\] (2.4)

Of particular interest in the present paper is the simplest superextension of the Virasoro algebra, the (centerless) Neveu–Schwarz algebra \(\mathfrak{N}\) which, apart from even basis elements \(L_n\), has odd basis elements \(G_r, r \in \frac{1}{2} + \mathbb{Z}\), with commutation relations
\[
[G_r, L_n] = \left( r - \frac{n}{2} \right) G_{r+n}, \quad [G_r, G_s] = 2 L_{r+s}.
\]
The algebra \(\mathfrak{N}\) is spanned by the formal distributions \(L(z) = \Sigma_n \mathbb{Z} L_n z^{-n-2}\) and \(G(z) = \Sigma_{r \in 1/2 + \mathbb{Z}} G_r z^{-r-3/2}\). The operator product expansions are given by [in addition to (2.4)]
\[
[L(z), G(w)] = \partial_z G(w) \delta(z-w) + \frac{1}{2} G(w) \partial_z \delta(z-w),
\] (2.5)
\[
[G(z), G(w)] = 2 L(w) \delta(z-w).
\]

Given a formal distribution Lie superalgebra \((g, \mathfrak{g})\), we may always include \(\mathfrak{g}\) in the minimal family \(\mathfrak{g}^c\) of pairwise local distributions which is closed under \(\partial_z\) and all products (2.3) (Ref. 3, Section 2.7). Then \(\mathfrak{g}^c\) is a conformal superalgebra with respect to the products (2.3) in the following sense:
A conformal superalgebra is a left \( \mathbb{Z}_2 \)-graded \( \mathbb{C}[\partial] \)-module \( R \) with a \( \mathbb{C} \)-bilinear product \( a_{(a)}b \) for each \( n \in \mathbb{Z}_+ \) such that the following axioms hold \([a,b,c] \in R; m,n \in \mathbb{Z}_+ \) and \( \partial^j = (1/j! \partial^j) \):

- \( (C0) \ a_{(a)}b = 0 \), for \( n \gg 0 \),
- \( (C1) \ (\partial a)_{(a)}b = -na_{(a-1)}b \),
- \( (C2) \ a_{(a)}b = (-1)^p(a)\sum_{j=0}^{\infty}(-1)^{j+1+n}(\partial^{j})(b_{(a+j)}a) \),
- \( (C3) \ a_{(m)}b_{(c)} = \sum_{j=0}^{\infty}(-1)^{m-j}a_{(j)}b_{(m-n+j)}c + (-1)^{p(a)p(b)}b_{(a)}(a_{(m)}c) \).

Note that \( (C1) \) and \( (C2) \) imply that \( a_{(a)}\partial b = \partial(a_{(a)}b) + na_{(a-1)}b \), and thus \( \partial \) is a derivation of all products \( (2.3) \).

Conversely, assuming for simplicity that \( R = \bigoplus_{i \in \mathbb{C}} \mathbb{C}[\partial]a^i \) is a free as a \( \mathbb{C}[\partial] \)-module conformal superalgebra, we may associate to \( R \) a formal distribution Lie superalgebra \( (g(R), \mathfrak{F}) \) with basis \( a_{(a)}^i(i \in I, m \in \mathbb{Z}) \) and \( \mathfrak{F} = \{ a^i(z) = \sum_{n \in \mathbb{N}_0} \partial^i[a^i]z^{n-1} \}_{i \in I} \) with bracket \( \text{cf. (2.2)} \):

\[
[a^i(z), a^j(w)] = \sum_{k \in \mathbb{Z}_+} (a^i_{(k)}a^j)(w) d^k_w \partial(z - w),
\]

so that \( \mathfrak{F} = R \). This formula is equivalent to the commutation relations \((m,n \in \mathbb{Z}; i,j \in I)\):

\[
[a^i_{(m)}, a^j_{(n)}] = \sum_{k \in \mathbb{Z}_+} \left( \frac{m}{k} \right)(a^i_{(k)}a^j)_{[m+n-k]}.
\]

The conformal algebra associated to the Virasoro algebra is the Virasoro conformal algebra \( R(\mathfrak{V}) = \mathbb{C}[\partial] \otimes L \) with products \( \text{cf. (2.4)} \)

\[
L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(ij)}L = 0, \quad \text{for } j > 1.
\]

The conformal superalgebra, associated to \( \mathfrak{N} \), is the Neveu–Schwarz conformal algebra \( R(\mathfrak{N}) = (\mathbb{C}[\partial] \otimes L) \oplus (\mathbb{C}[\partial] \otimes G) \) with additional nonzero products \( \text{cf. (2.5)} \):

\[
L_{(0)}G = \partial G, \quad G_{(1)}L = \frac{1}{2} \partial G, \quad L_{(1)}G = G_{(1)}L = G(0)L = 2L.
\]

### B. Conformal modules

Let \( (g, \mathfrak{F}) \) be a formal distribution Lie superalgebra. Let \( V \) be a \( g \)-module and suppose that \( V \) is spanned over \( \mathbb{C} \) by the coefficients of a family \( \mathcal{E} \) of formal distributions such that all \( a(z) \in \mathfrak{F} \) are local with respect to all \( v(z) \in \mathcal{E} \). Then we call \((V, \mathcal{E}) \) a conformal module over \((g, \mathfrak{F}) \).

**Example 2.1:** \( \mathfrak{V} \) of course is the Lie algebra of regular vector fields on \( \mathbb{C}^\times \), where \( L_n = -t^{n+1}dt/dt, n \in \mathbb{Z} \). For \( \alpha, \Delta \in \mathbb{C} \) let

\[
F_{\mathfrak{V}}(\alpha, \Delta) = C[t, t^{-1}] e^{-at}dt^{1-\Delta}.
\]

The Lie algebra \( \mathfrak{V} \) acts on the space \( F_{\mathfrak{V}}(\alpha, \Delta) \) in a natural way:

\[
\left( f(t) \frac{\partial}{\partial t} \right) g(t) dt^{1-\Delta} = (f(t)g'(t) + (1-\Delta)g(t)f'(t)) dt^{1-\Delta}, \tag{2.6}
\]

where \( f(t) \in C[t, t^{-1}] \) and \( g(t) \in C[t, t^{-1}] e^{-at} \). Letting \( v_{[a]} = t^n e^{-at} dt^{1-\Delta} \) and \( v(z) = \sum_{n \in \mathbb{Z}} z^n z^{-n-1} \), (2.6) is equivalent to

\[
L(z)v(w) = (\partial_w + \alpha)v(w) \partial(z-w) + \Delta v(w) \partial_w \partial(z-w).
\]

Hence this module is conformal.

**Example 2.2:** Similarly \( \mathfrak{N} \) is a subalgebra of the Lie superalgebra of regular vector fields on \( \mathbb{C}^\times \) with \( N = 1 \) extended symmetry described below. Let \( \xi \) be the odd indeterminate and \( W(1, 1) \) be the Lie superalgebra of regular vector fields on the circle with \( N = 1 \) extended symmetry. Then \( \mathfrak{N} \) is spanned by \((n \in \mathbb{Z} \text{ and } r \in \mathbb{Z}/2) \)
Equivalently, consider the contact form
\[ \omega = dt - \xi d\xi. \]

Then \( \mathfrak{M} = \{ D \in W(1,1)| D \omega = f_D \omega, \) for some \( f_D \in C[t, t^{-1}, \xi] \}. \) It is well known (see, e.g., Ref. 4) that every element \( D \in \mathfrak{M} \) is of the form
\[ D^u = 2u \frac{\partial}{\partial t} + (-1)^{p(u)} \left( \xi \frac{\partial u}{\partial t} + \xi \frac{\partial u}{\partial \xi} \right), \]
with \( u \in C[t, t^{-1}, \xi] \). One has \( D^u \omega = 2(\partial u/\partial t) \omega \). As in (2.6), \( \mathfrak{M} \) acts on the space \( (\alpha, \Delta \in C) \)
\[ F_{\mathfrak{M}}(\alpha, \Delta) = C[t, t^{-1}, \xi]e^{-at} \omega^{1-\Delta}, \]
in a natural way, via \( (f \in C[t, t^{-1}, \xi]e^{-at}) \):
\[ D^u(f \omega^{1-\Delta}) = D^u(f) \omega^{1-\Delta} + (-1)^{p(f) p(u)}(1-\Delta) f \omega^{-\Delta} D^u(\omega), \]
which is equivalent to
\[ D^u(f \omega^{1-\Delta}) = \left( D^u(f) + 2(1-\Delta) \frac{\partial u}{\partial t} f \right) \omega^{1-\Delta}. \quad (2.7) \]

Let \( v(z) = \sum_{n \in \mathbb{Z}} a_n e^{-at} \omega^{1-\Delta} z^{-n-1} \) and \( v^\xi(z) = \sum_{n \in \mathbb{Z}} \xi e^{-at} \omega^{1-\Delta} z^{-n-1} \). Then (2.7) gives
\[ L(z) v(w) = (\partial w + \alpha) v(w) \delta(z-w) + \Delta v(w) \partial w \delta(z-w), \]
\[ L(z) v^\xi(w) = (\partial w + \alpha) v^\xi(w) \delta(z-w) + (\Delta - \frac{1}{2}) v^\xi(w) \partial w \delta(z-w), \]
\[ G(z) v(w) = (\partial w + \alpha) v^\xi(w) \delta(z-w) + (2\Delta - 1) v^\xi(w) \partial w \delta(z-w), \]
\[ G(z) v^\xi(w) = v(w) \delta(z-w). \]

The parity on \( F_{\mathfrak{M}}(\alpha, \Delta) \) is defined in a natural way: \( p(v(z)) = p(v^\xi(z)) + 1 = 0 \). Reversing the parity we obtain a module, denoted by \( F_{\mathfrak{M}}(\alpha, \Delta - \frac{1}{2}) \). Obviously both modules are conformal.

One can show that the family \( \mathcal{E} \) of a conformal module \( (V, \mathcal{E}) \) over \( (g, \mathfrak{g}) \) can always be included in a larger family \( \mathcal{E}^\ast \), which is still local with respect to \( \mathfrak{g} \), and such that \( \partial \mathcal{E} \subset \mathcal{E} \) and \( a_{(j)} \mathcal{E} \subset \mathcal{E} \) for all \( a \in \mathfrak{g} \) and \( j \in \mathbb{Z}_+ \). One checks that for \( a, b \in \mathfrak{g} \) and \( v \in \mathcal{E}(m, n, \mathbb{Z}_+) \):
\[ [a_{(m)}, b_{(n)}] v = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (a_{(j)} b_{(m+n-j)}) v, \quad (\partial a_{(n)} v = [\partial, a_{(n)}]) v = -n a_{(n-1)} v. \]

Therefore, it follows that any conformal module \( (V, \mathcal{E}) \) over a formal distribution Lie superalgebra \( (g, \mathfrak{g}) \) gives rise to a module \( M = \mathcal{E} \) over the conformal algebra \( R = \mathfrak{g} \), defined as follows.13 It is a (left) \( \mathbb{Z}_2 \)-graded \( C[\partial] \)-module equipped with a family of \( C \)-linear maps \( a_{(n)}^M \) of \( R \) to \( \text{End}_R M \), for each \( n \in \mathbb{Z}_+ \), such that the following properties hold for \( a, b \in R \) and \( m, n \in \mathbb{Z}_+ \):

(M0) \( a_{(n)}^M v = 0, \) for \( v \in M \) and \( n \gg 0, \)

(M1) \[ [a_{(m)}^M, b_{(n)}^M] = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) (a_{(j)} b_{(m+n-j)})^M, \]

(M2) \( (\partial a_{(n)}^M = [\partial, a_{(n)}])^M = -n a_{(n-1)}^M. \)
Conversely, suppose that a conformal superalgebra \( R = \oplus_{i \in J} \mathbb{C}[\partial]a^i \) is a free \( \mathbb{C}[\partial] \)-module and consider the associated formal distribution Lie superalgebra \( (g(R), \mathfrak{z}) \). Let \( M \) be a module over the conformal algebra \( R \) and suppose that \( M \) is a free \( \mathbb{C}[\partial] \)-module with \( \mathbb{C}[\partial] \)-basis \( \{v^\alpha\}_{\alpha \in I} \). This gives rise to a conformal module \( M^c \) over \( g(R) \) with basis \( v^\alpha_{[n]} \), where \( i \in I \), \( \alpha \in J \), and \( n \in \mathbb{Z} \), defined by

\[
a^i(z)v^\alpha(w) = \sum_{j \in \mathbb{Z}^+} (a^i_{(j)}v^\alpha)(w)\partial^j_w \delta(z - w).
\]

In the general case one defines \( M^c \) as the quotient of the space \( \oplus_{j \in \mathbb{Z}} M_{[j]} \), where \( M_{[j]} \) is a copy of \( M \), by the \( \mathbb{C} \)-span of \( \{(\partial \partial^j)v^\alpha_{[n]} + \nu v^\alpha_{[n-1]} | v \in M, \ n \in \mathbb{Z}\} \), and let \( \mathcal{E} = \left\{ v(z) = \sum_{n \in \mathbb{N}} v^\alpha_{[n]}z^{-n-1} | v \in M \right\} \). The conformal module \( (M^c, \mathcal{E}) \) is called the maximal conformal module over \( (g(R), \mathfrak{z}) \) associated to the \( R \)-module \( M \). The correspondence which associates to an \( R \)-module \( M \) the collection of quotients of the module \( (M^c, \mathcal{E}) \) by submodules having a trivial intersection with \( \mathcal{E} \) is bijective. \(^3\)

The above discussion reduces the classification of finite conformal modules over a formal distribution Lie superalgebra \( (g, \mathfrak{z}) \) to the classification of finite modules over the corresponding conformal superalgebra.

Introducing the generating series

\[
a_{\alpha\beta} = \sum_{n=0}^\infty a_{(n)} b_{\alpha\beta} \frac{\lambda^n}{n!} \quad \text{and} \quad a_{\alpha\beta}^M = \sum_{n=0}^\infty a_{(n)}^M \frac{\lambda^n}{n!},
\]

which lie in \( \mathbb{C}[\lambda] \otimes \mathcal{R} \) and \( \mathbb{C}[\lambda] \otimes \text{End}_{R}(M) \) due to (C0) and (M0), respectively, identities (M1) and (M2) can be written as

\[
\left[ a_{\alpha\beta}^M, b_{\mu\nu}^M \right] = (a_{\alpha\beta} b_{\mu\nu})^M_{\lambda+\mu} \quad \text{and} \quad \left[ \partial a_{\alpha}^M, a_{\beta}^M \right] = -\lambda a_{\lambda}^M. \tag{2.8}
\]

Let \( M \) be a \( \mathbb{C}[\partial] \)-module. The following is immediate from (2.8).

(a) An \( R(\mathfrak{W}) \)-module structure on \( M \) is given by \( L^M_{\lambda} \in \mathbb{C}[\lambda] \otimes \text{End}_{C}(M) \) such that

\[
\left[ L^M_{\lambda}, L^M_{\mu} \right] = (\lambda - \mu)L^M_{\lambda+\mu}, \tag{2.9}
\]

\[
\left[ \partial, L^M_{\lambda} \right] = -\lambda L^M_{\lambda}. \tag{2.10}
\]

(b) An \( R(\mathfrak{Z}) \)-module structure is given by \( L^M_{\lambda}, G^M_{\mu} \in \mathbb{C}[\lambda] \otimes \text{End}_{C}(M) \) satisfying in addition to (2.9) and (2.10)

\[
\left[ L^M_{\lambda}, G^M_{\mu} \right] = \frac{\lambda}{2} - \mu G^M_{\lambda+\mu}, \tag{2.11}
\]

\[
\left[ G^M_{\lambda}, G^M_{\mu} \right] = 2L^M_{\lambda+\mu}, \tag{2.12}
\]

\[
\left[ \partial, G^M_{\lambda} \right] = -\lambda G^M_{\lambda}. \tag{2.13}
\]

A conformal module \( (V, \mathcal{E}) \) (respectively module \( M \)) over a formal distribution Lie superalgebra \( (g, \mathfrak{z}) \) (respectively over a conformal superalgebra \( R \)) is called finite, if \( \mathcal{E} \) (respectively \( M \)) is a finitely generated \( \mathbb{C}[\partial] \)-module.

A conformal module \( (V, \mathcal{E}) \) over \( (g, \mathfrak{z}) \) is called irreducible if there is no nontrivial invariant subspace which contains all \( v^\alpha_{[n]}, n \in \mathbb{Z} \), for some nonzero \( v \in \mathcal{E} \). Clearly a conformal module is irreducible if and only if the associated module \( \mathcal{E} \) over the conformal superalgebra \( \mathfrak{z} \) is irreducible.

Example 2.3: Example 2.1 gives a family of \( R(\mathfrak{Z}) \)-modules \( V(\alpha, \Delta) = \mathbb{C}[\partial] \otimes v_{\Delta} \) with

\[
L^\lambda v_{\Delta} = (\partial + \alpha + \Delta \lambda)v_{\Delta}.
\]
Example 2.4: Example 2.2 gives two families of $R(\mathfrak{I})$-modules. Namely, $F'_n(\alpha, \Delta)$ gives rise to $N'(\alpha, \Delta) = C[\partial]v_\Delta + C[\partial]v_\Delta^\epsilon$ with $p(v_\Delta) = p(v_\Delta^\epsilon) + \tilde{\epsilon} = 0$ and
\begin{equation}
L_\lambda v_\Delta = (\partial + \alpha + \Delta \lambda)v_\Delta, \quad L_\lambda v_\Delta^\epsilon = (\partial + \alpha + (\Delta - \frac{1}{2})\lambda)v_\Delta^\epsilon, \quad G_\lambda v_\Delta = (\partial + \alpha + (2\Delta - 1)\lambda)v_\Delta, \quad G_\lambda v_\Delta^\epsilon = v_\Delta.
\end{equation}
The module $F_n(\alpha, \Delta)$ gives rise to $N(\alpha, \Delta) = C[\partial]v_\Delta + C[\partial]v_\Delta^\epsilon$ with $p(v_\Delta) = p(v_\Delta^\epsilon) + \tilde{\epsilon} = 0$ and
\begin{equation}
L_\lambda v_\Delta = (\partial + \alpha + \Delta \lambda)v_\Delta, \quad L_\lambda v_\Delta^\epsilon = (\partial + \alpha + (\Delta + \frac{1}{2})\lambda)v_\Delta^\epsilon, \quad G_\lambda v_\Delta = v_\Delta^\epsilon, \quad G_\lambda v_\Delta^\epsilon = (\partial + \alpha + 2\Delta \lambda)v_\Delta.
\end{equation}
We note that the notation we have chosen here for the modules $N(\alpha, \Delta)$ and $N'(\alpha, \Delta)$ is such that the “conformal weight” of the corresponding even vector $v_\Delta$ is $\Delta$. Therefore, reversing the parity of $N'(\alpha, \Delta)$ does not give $N(\alpha, \Delta)$, but rather $N(\alpha, \Delta - \frac{1}{2})$.

Theorem 2.1: The following is a complete list of non-one-dimensional (over $C$) finite irreducible modules over the conformal superalgebras $R(\mathfrak{V})$ and $R(\mathfrak{I})$:

(a) $V(\alpha, \Delta)$, where $\alpha, \Delta \in C$ with $\Delta \neq 0$, and
(b) $N(\alpha, \Delta)$ and $N'(\alpha, \Delta')$, where $\alpha, \Delta, \Delta' \in C$ with $\Delta \neq 0$ and $\Delta' \neq \frac{1}{2}$.

C. Extensions of Virasoro conformal modules

We first review the notion of an extension. Given two modules $V$ and $W$ over a conformal superalgebra (or a Lie superalgebra) $R$, an exact sequence of $R$-modules of the form
\[
0 \rightarrow V \xrightarrow{i} F \xrightarrow{p} W \rightarrow 0
\]
is called an extension of $W$ by $V$. Two extensions
\[
0 \xrightarrow{i} V \xrightarrow{p} W \rightarrow 0 \quad \text{and} \quad 0 \xrightarrow{i'} V \xrightarrow{p'} W \rightarrow 0
\]
are said to be equivalent if there exists a commutative diagram of the form
\[
\begin{array}{ccc}
0 & \xrightarrow{i} & V \\
& \downarrow & \psi \\
& & W \\
0 & \xrightarrow{i'} & V \\
& \downarrow & 1_W
\end{array}
\]
where $1_V : V \rightarrow V$ and $1_W : W \rightarrow W$ are the respective identity maps and $\psi : F \rightarrow F'$ is a homomorphism of modules.

The direct sum of modules $V \oplus W$ obviously gives rise to an extension. Extensions equivalent to it are called trivial extensions. In general an extension can be thought of as the direct sum of vector spaces $E = V \oplus W$, where $V$ is a submodule of $E$, while for $w$ in $W$ we have
\[
a \cdot w = aw + \phi_a(w), \quad a \in R,
\]
where $\phi_a : W \rightarrow V$ is a linear map satisfying the cocycle condition: $\phi_{[a, b]}(w) = \phi_a(bw) + a\phi_b(w) - (-1)^{p(a)p(b)}(\phi_b(aw) + b\phi_a(w)), b \in R$. The set of these cocycles form a vector space over $C$. Cocycles corresponding to the trivial extension are called trivial cocycles. They form a subspace and the dimension of the quotient space by it is called the dimension of the space of extensions of $W$ by $V$. 

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According to Theorem 2.1 every finite non-one-dimensional (over \( \mathbb{C} \)) irreducible module over the Virasoro conformal algebra \( R(\mathfrak{V}) \) is of the form \( V(\alpha, \Delta) \) with \( \Delta \neq 0 \). Let \( \mathcal{C}_a, \alpha \in \mathbb{C} \), denote the one-dimensional vector space over \( \mathbb{C} \) on which we have an action of \( R(\mathfrak{V}) \) defined by

\[
L_\lambda c_a = 0, \quad \partial c_a = \alpha c_a.
\]

It is easy to see that \( \mathcal{C}_a \) are the only one-dimensional modules over \( R(\mathfrak{V}) \). Thus \( V(\alpha, \Delta) \) with \( \Delta \neq 0 \), together with \( \mathcal{C}_a \), form a complete list of finite irreducible modules over the Virasoro conformal algebra.

In Ref. 2 extensions over \( R(\mathfrak{V}) \) of the following types have been classified \( (\Delta, \Delta' \neq 0) \):

\[
0 \rightarrow \mathcal{C}_a \rightarrow E \rightarrow V(\beta, \Delta) \rightarrow 0,
\]

\[
0 \rightarrow V(\alpha, \Delta) \rightarrow E \rightarrow \mathcal{C}_\beta \rightarrow 0,
\]

and

\[
0 \rightarrow V(\alpha, \Delta') \rightarrow E \rightarrow V(\beta, \Delta) \rightarrow 0.
\]

As a module over \( \mathbb{C} [\partial] \), \( E \) in (2.16) is isomorphic to \( \mathcal{C}_a \oplus V(\beta, \Delta) \), where \( \mathcal{C}_a \) is an \( R(\mathfrak{V}) \)-submodule, and \( V(\alpha, \Delta) = \mathbb{C} [\partial] v_\Delta \) with

\[
L_\lambda v_\Delta = (\partial + \beta + \Delta \lambda)v_\Delta + f(\lambda) c_a, \quad f(\lambda) \in \mathbb{C} [\lambda].
\]

**Theorem 2.2:** Nontrivial extensions of the form (2.16) exist if and only if \( \alpha + \beta = 0 \) and \( \Delta = 1, 2 \). In these cases they are given, up to equivalence, by (2.19), where

(i) \( f(\lambda) = a_2 \lambda^2 \), for \( \Delta = 1 \) and \( a_2 \neq 0 \),

(ii) \( f(\lambda) = a_3 \lambda^3 \), for \( \Delta = 2 \) and \( a_3 \neq 0 \).

Furthermore, all trivial cocycles are given by scalar multiples of the polynomial \( f(\lambda) = \alpha + \beta + \Delta \lambda \).

**Proof:** That (i) and (ii) are the only nontrivial extensions of the form (2.16) is proved in Ref. 2. Here we will concentrate on finding the formula of the trivial cocycles. We will omit similar calculations in the future.

Suppose (2.19) represents a trivial cocycle. This means that the exact sequence (2.16) is split and hence there exists \( v' = g(\partial)v_\Delta + bc_a \in E \), where \( b \in \mathbb{C} \), such that

\[
L_\lambda v' = (\partial + \beta + \Delta \lambda)v',
\]

which gives

\[
L_\lambda v' = (\partial + \beta + \lambda)g(\partial)v_\Delta + b(\partial + \beta + \lambda)c_a.
\]

It follows from (2.10) and (2.19) that

\[
L_\lambda v' = g(\partial + \lambda)(\partial + \beta + \lambda)v_\Delta + g(\partial + \lambda)f(\lambda)c_a.
\]

Comparing both expressions for \( L_\lambda v' \), we see that \( g \) is a constant and \( f(\lambda) \) is a scalar multiple of \( \partial + \beta + \Delta \lambda \).

**Remark 2.1:** In what follows we shall often employ a shift by \( \alpha \in \mathbb{C} \) of an equation \( E = 0 \) on \( f(\partial) \), which amounts to introducing a new indeterminate \( \bar{\partial} = \partial + \alpha \) and a new function \( \bar{f}(\bar{\partial}) = f(\bar{\partial} - \alpha) \). Denote by \( \bar{E} = 0 \) the resulting equation. Obviously, if \( f(\bar{\partial}) \) is a solution of \( \bar{E} = 0 \), then \( f(\partial + \alpha) \) is a solution of \( E = 0 \) and vice versa. We shall write \( \bar{f} \) in place of \( \bar{f} \) and \( \bar{\partial} \) in place of \( \bar{\partial} \).

As a vector space, \( E \) in (2.17) is isomorphic to \( V(\alpha, \Delta) \oplus \mathcal{C}_\beta \). Here \( V(\alpha, \Delta) \) is an \( R(\mathfrak{V}) \)-submodule and we have
where \( f(\partial,\lambda) \in \mathbb{C} [\partial,\lambda] \) and \( a(\partial) \in \mathbb{C} [\partial] \). Applying (2.9) to \( \nu_\Delta \) and applying a shift by \( \alpha \) gives the identity

\[
(\partial + \Delta \lambda)f(\partial + \lambda, \mu) - (\partial + \Delta \mu)f(\partial + \mu, \lambda) = (\lambda - \mu)f(\partial, \lambda + \mu),
\]

while applying (2.10) gives

\[
(\partial + \Delta \lambda)\alpha(\partial + \lambda) = (\partial - (\alpha + \beta) + \lambda)f(\partial, \lambda).
\]

Solutions to (2.21) and (2.22), corresponding to nontrivial extensions, are given by the following theorem.

**Theorem 2.3**: Nontrivial extensions of the form (2.17) exist if and only if \( \alpha + \beta = 0 \) and \( \Delta = 1 \). These extensions are given, up to equivalence, by (2.20), where \( f(\partial, \lambda) = \alpha(\partial) = a_0, \ a_0 \neq 0 \). Furthermore, all trivial cocycles correspond to pairs of the form \( f(\partial, \lambda) = (\partial + \alpha + \Delta \lambda)h(\partial + \lambda) \) and \( \alpha(\partial) = (\partial - \beta)h(\partial) \), where \( h \) is any polynomial.

Let \( E \) be an extension of the form (2.18). As a \( \mathbb{C} [\partial] \)-module, \( E = \mathbb{C} [\partial]v_\Delta \oplus \mathbb{C} [\partial]v_\Delta' \), where \( \mathbb{C} [\partial]v_\Delta \) is an \( R(\mathfrak{g}) \)-submodule of \( E \). We have

\[
L_\lambda v_\Delta = (\partial + \beta + \Delta \lambda)v_\Delta + f(\partial, \lambda)v_\Delta',
\]

for some polynomial \( f(\partial, \lambda) \).

**Theorem 2.4**: (cf. Ref. 6) Nontrivial extensions of the form (2.18) exist only if \( \alpha = \beta \) and \( \Delta - \Delta' = 0,1,2,3,4,5,6 \). The following is a complete list of values of \( \Delta \) and \( \Delta' \) along with the corresponding polynomials \( f(\partial, \lambda) \), whose nonzero scalar multiples give rise to nontrivial extensions \( \tilde{\alpha} = \partial + \alpha \):

\[
\begin{align*}
(i) & \quad \Delta = \Delta', \ f(\partial, \lambda) = r\lambda + s,(r,s) \neq (0,0). \\
(ii) & \quad \Delta = 1, \Delta' = 0, \ f(\partial, \lambda) = r\lambda + s\lambda^2 + t\lambda, \ (r,s,t) \neq (0,0,0). \\
(iii) & \quad \Delta = \Delta' = 2, \ f(\partial, \lambda) = \lambda^2(2\partial + \lambda). \\
(iv) & \quad \Delta = \Delta' = 3, \ f(\partial, \lambda) = \lambda^2(3\partial + \lambda). \\
(v) & \quad \Delta = \Delta' = 4, \ f(\partial, \lambda) = \lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \Delta'\lambda^3). \\
(vi) & \quad \Delta = 5, \Delta' = 0, \ f(\partial, \lambda) = 5\lambda^5 + 10\partial^2\lambda^4 - \partial\lambda^5. \\
(vi') & \quad \Delta = 1, \Delta' = -4, \ f(\partial, \lambda) = \lambda^2\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6. \\
(vii) & \quad \Delta = \frac{7}{2} \pm \sqrt{19}/2, \Delta - \Delta' = 6, \ f(\partial, \lambda) = \lambda^3(\partial^5 + 6\partial^4\lambda - \partial^3\lambda^2 + \Delta'\lambda^3) - (3\Delta' + 1)\partial^3\lambda^6 - (\Delta' + 2)\partial\lambda^7.
\end{align*}
\]

Furthermore, all trivial cocycles are given by \( f(\partial, \lambda) = (\partial + \alpha + \Delta'\lambda)h(\partial + \lambda) - (\partial + \beta + \Delta\lambda)h(\partial) \), where \( h \) is any polynomial.

### III. Extensions Involving One-Dimensional Modules

Let \( \alpha \in \mathbb{C} \) and let \( Ce_\alpha \) be the even one-dimensional module over \( R(\mathfrak{g}) \) given by

\[
L_\lambda e_\alpha = G_\lambda e_\alpha = 0, \quad \partial e_\alpha = \alpha e_\alpha.
\]

Reversing the parity we obtain the \( R(\mathfrak{g}) \)-module \( Ce_\alpha \). In this section we will consider extensions between \( N(\beta, \Delta) \) and \( Ce_\alpha \) and between \( N(\beta, \Delta) \) and \( Ce_\alpha \). From our classification of such extensions, extensions between \( N'(\beta, \Delta) \) and \( Ce_\alpha \) and between \( N'(\beta, \Delta) \) and \( Ce_\alpha \) are then obtained by reversing parity.

Let

\[
0 \rightarrow Ce_\alpha \rightarrow E \rightarrow N(\beta, \Delta) \rightarrow 0,
\]
be an exact sequence of $R(\mathfrak{g})$-modules. As a $C[\partial]$-module we have $E\cong C_{c_\alpha}\oplus N(\beta,\Delta)$, where $N(\beta,\Delta)\cong C[\partial]v_\Delta \oplus C[\partial]v_\Delta^\perp$. $C_{c_\alpha}$ is an $R(\mathfrak{g})$-submodule of $E$, and the action of $R(\mathfrak{g})$ is given by

$$L_\lambda v_\Delta = (\partial + \beta + \Delta \lambda)v_\Delta + f(\lambda)c_\alpha, \quad L_\lambda v_\Delta^\perp = (\partial + \beta + (\Delta + \frac{1}{2})\lambda)v_\Delta^\perp,$$

$$G_\lambda v_\Delta = v_\Delta^\perp, \quad G_\lambda v_\Delta^\perp = (\partial + \beta + 2\Delta \lambda)v_\Delta + g(\lambda)c_\alpha,$$

(3.2)

where $f(\lambda), g(\lambda) \in C[\lambda]$.

The following lemma is straightforward.

**Lemma 3.1:** All trivial extensions of the form (3.1) are of the form (3.2), where $f(\lambda)$ and $g(\lambda)$ are scalar multiples (by the same scalar) of $\alpha + \beta + \Delta \lambda$ and $\alpha + \beta + 2\Delta \lambda$, respectively.

Applying (2.12) to the vector $v_\Delta$ we obtain

$$g(\lambda) + g(\mu) = 2f(\lambda + \mu).$$

(3.3)

Putting $\mu = 0$ in (3.3) we get

$$f(\lambda) = \frac{1}{2}g(\lambda) + \text{const},$$

from which we obtain, plugging this back into (3.3),

$$f(\lambda) + f(\mu) = f(\lambda + \mu) + \text{const}.$$

Hence $f(\lambda)$ is a linear function, from which we conclude, using Theorem 2.2, that $f(\lambda)$ must correspond to the trivial cocycle. But then by Lemma 3.1 we may assume from the very beginning that $f(\lambda) = 0$. But then (3.3) implies that $g(\lambda) = 0$ and so the extension is equivalent to the trivial extension. Thus we have proved the following.

**Proposition 3.1:** There are no nontrivial extensions of the form (3.1).

Consider extensions of $R(\mathfrak{g})$-modules of the form

$$0 \rightarrow C_{c_\alpha} \rightarrow E \rightarrow N(\beta,\Delta) \rightarrow 0.$$

(3.4)

As a $C[\partial]$-module we have $E\cong C_{c_\alpha}\oplus N(\beta,\Delta)$, where $N(\beta,\Delta)\cong C[\partial]v_\Delta \oplus C[\partial]v_\Delta^\perp$. We have

$$L_\lambda v_\Delta = (\partial + \beta + \Delta \lambda)v_\Delta, \quad L_\lambda v_\Delta^\perp = (\partial + \beta + (\Delta + \frac{1}{2})\lambda)v_\Delta^\perp + f(\lambda)c_\alpha,$$

$$G_\lambda v_\Delta = v_\Delta^\perp + g(\lambda)c_\alpha, \quad G_\lambda v_\Delta^\perp = (\partial + \beta + 2\Delta \lambda)v_\Delta,$$

(3.5)

where $f(\lambda), g(\lambda) \in C[\lambda]$.

Again we first compute trivial cocycles corresponding to extensions of the form (3.4).

**Lemma 2.2:** All trivial extensions of the form (3.4) correspond to scalar multiples of the pair $f(\lambda) = \alpha + \beta + (\Delta + \frac{1}{2})\lambda$ and $g(\lambda) = 1$ in (3.5).

Applying (2.9) to the vector $v_\Delta^\perp$ we obtain

$$(\alpha + \beta + (\Delta + \frac{1}{2})\mu + \lambda)f(\lambda) - (\alpha + \beta + (\Delta + \frac{1}{2})\lambda + \mu)f(\mu) = (\lambda - \mu)f(\lambda + \mu),$$

(3.6)

while applying (2.11) to the vector $v_\Delta$ we get

$$f(\lambda) - (\alpha + \beta + \mu + \Delta \lambda)g(\mu) = \left(\frac{\lambda}{2} - \mu\right)g(\lambda + \mu).$$

(3.7)

Putting $\mu = 0$ in (3.6) we obtain, if $\alpha + \beta \neq 0$, that

$$f(\lambda) = \left(\frac{f(0)}{\alpha + \beta}\right)\left(\alpha + \beta + \left(\Delta + \frac{1}{2}\right)\lambda\right).$$
Hence by Lemma 3.2 we may subtract a trivial cocycle and assume that \( f(\lambda) = 0 \). Now if \( f(\lambda) = 0 \), we let \( \mu = 0 \) in (3.7) and get \( g(\lambda) = a_0 \) is a constant. Furthermore, if \( g(\lambda) \neq 0 \), then (3.7) gives \( \Delta = -\frac{1}{2} \) and \( \alpha + \beta = 0 \). But then Lemma 3.2 implies that such an extension is a trivial extension.

Hence we may assume that \( \alpha + \beta = 0 \) and \( f(\lambda) = \lambda^3 \) and \( \Delta = \frac{1}{2} \), or else \( f(\lambda) = \lambda^2 \) and \( \Delta = \frac{1}{2} \) by Theorem 2.2. Since the equation (3.7) is homogeneous, we see that \( g(\lambda) = a\lambda^2 \) in the first case and \( g(\lambda) = a\lambda \) in the second case \( (a \in \mathbb{C}) \). Setting \( \mu = 0 \) in (3.7) we get \( 2f(\lambda) = \lambda g(\lambda) \), so that we end up with the following two nontrivial solutions:

**Theorem 3.1**: Extensions of the form (3.4) exist if and only if \( \alpha + \beta = 0 \) and \( \Delta \) is either \( \frac{1}{2} \) or \( \frac{1}{2} \). In these cases they are given, up to equivalence, by

1. \( f(\lambda) = a_3\lambda^3 \), \( g(\lambda) = 2a_3\lambda^2 \) in the case \( \Delta = \frac{1}{2} \) where \( a_3 \neq 0 \); and
2. \( f(\lambda) = a_2\lambda^2 \), \( g(\lambda) = 2a_2\lambda \) in the case \( \Delta = \frac{1}{2} \) where \( a_2 \neq 0 \).

**Remark 3.1**: As conformal modules these two extensions are as follows: It is more convenient to reverse the parity. Let \( f(t) \in \mathbb{C}[t, t^{-1}] \), \( g(t) \in \mathbb{C}[t, t^{-1}]e^{-at} \), and \( c_{-a} \) be an even vector. Theorem 3.1(i) corresponds to the module \( \mathbb{C}[t, t^{-1}, \xi]e^{-at} \omega^{-1} \oplus \mathbb{C}c_{-a} \) with action of \( \mathfrak{h} \) given by

\[
D^{(t)}g(t)\omega^{-1} = (f(t)g'(t) - f'(t)g(t))\omega^{-1} + \text{Res}_{t=0}(f'''(t)g(t))c_{-a},
\]
\[
D^{(t)}g(t)\xi\omega^{-1} = (f(t)g'(t) - \frac{f'(t)g(t)}{2})\xi\omega^{-1},
\]
\[
D^{(t)}g(t)\xi\omega^{-1} = (f(t)g'(t) - 2f'(t)g(t))\xi\omega^{-1},
\]
\[
D^{(t)}g(t)\xi\omega^{-1} = -f(t)g(t)\omega^{-1} + 2\text{Res}_{t=0}(f'''(t)g(t))c_{-a}.
\]

while Theorem 3.1(ii) corresponds to the vector space \( \mathbb{C}[t, t^{-1}, \xi]e^{-at} \oplus \mathbb{C}c_{a} \) with action

\[
D^{(t)}g(t) = f(t)g'(t) + \text{Res}_{t=0}(f'''(t)g(t))c_{-a},
\]
\[
D^{(t)}g(t)\xi = \left( f(t)g'(t) + \frac{f'(t)g(t)}{2} \right)\xi,
\]
\[
D^{(t)}g(t)\xi = f(t)g'(t)\xi,
\]
\[
D^{(t)}g(t)\xi = -f(t)g(t) + 2\text{Res}_{t=0}(f'(t)g(t))c_{-a}.
\]

These two extensions of course are extensions of the adjoint module and the module of functions by an even trivial module. Note that the conformal module, corresponding to the \( R(\mathfrak{g}) \)-module of Theorem 3.1(i) in the case when \( \alpha = 0 \), is isomorphic, as an \( \mathfrak{g} \)-module, to the unique central extension of \( \mathfrak{g} \).

Next consider extensions of the form

\[
0 \rightarrow N(\alpha, \Delta) \rightarrow E \rightarrow \mathbb{C}c_{\beta} \rightarrow 0.
\]

As a vector space we have \( E = N(\alpha, \Delta) \oplus \mathbb{C}c_{\beta} \), where \( N(\alpha, \Delta) \) is an \( R(\mathfrak{g}) \)-submodule and the action on \( c_{\beta} \) is given by

\[
L_{\lambda}c_{\beta} = f(\partial, \lambda)v_{\Delta}, \quad G_{\lambda}c_{\beta} = g(\partial, \lambda)v_{\Delta}, \quad \partial c_{\beta} = \beta c_{\beta} + a(\partial)v_{\Delta}.
\]

**Lemma 3.3**: All trivial extensions of the form (3.8) are given by (3.9) with triples \( f(\partial, \lambda), \)
\( g(\partial, \lambda), \) and \( a(\partial) \) of the form \( (\partial + \alpha + \Delta \lambda)h(\partial + \lambda), h(\partial + \lambda), \) and \( (\partial - \beta)h(\partial) \), respectively, where \( h \) is a polynomial.
Applying (2.9) to the vector $c_{\beta}$ we obtain (2.21), while applying (2.10) and (2.13) gives (2.22)

$$a(\partial + \lambda) = (\partial + \lambda - \beta)g(\partial, \lambda),$$

(3.10)

respectively. If $a = 0$, we have $f = 0$ and $g = 0$ by (2.22) and (3.10), respectively. Hence we may assume that $a \neq 0$. Combining (2.22) and (3.10) we obtain

$$f(\partial, \lambda) = g(\partial, \lambda)(\partial + \alpha + \Delta \lambda).$$

(3.11)

Putting $\mu = 0$ in (2.21) and we get, after a shift by $\alpha$ (see Remark 2.1),

$$(\partial + \Delta \lambda)f(\partial + \lambda, 0) = (\partial + \lambda)f(\partial, \lambda).$$

We may assume that $f$ is homogeneous in $\partial$ and $\lambda$ and consider first the case when $f$ is not a constant. Then

$$f(\partial, \lambda) = (\partial + \Delta \lambda)h(\partial + \lambda),$$

(3.12)

where $h(\partial + \lambda) = [f(\partial + \lambda, 0)]/(\partial + \lambda)$ is a polynomial. Substituting (3.12) into (2.22) and (3.11) we obtain, respectively,

$$a(\partial + \lambda) = (\partial + \lambda - (\alpha + \beta))h(\partial + \lambda),$$

(3.13)

$$g(\partial, \lambda) = h(\partial + \lambda).$$

(3.14)

However, this is the trivial extension by Lemma 3.3.

Thus we may assume that $f = 1$ and $\Delta = 1$ by Theorem 2.3. But then (2.22) implies that $\alpha + \beta = 0$ and $a = 1$. But this contradicts (3.10). Hence we have proved the following.

**Proposition 3.2:** There are no nontrivial extensions of the form (3.8).

Finally consider extensions of the form

$$0 \to N(\alpha, \Delta) \to E \to \mathbb{C} \epsilon_{\beta} \to 0.$$  

(3.15)

As a vector space we have $E = N(\alpha, \Delta) \oplus \mathbb{C} \epsilon_{\beta}$ with $R(\mathfrak{h})$-submodule $N(\alpha, \Delta)$ and

$$L_{\lambda} \epsilon_{\beta} = f(\partial, \lambda)v_{\Delta}^\beta, \quad G_{\lambda} \epsilon_{\beta} = g(\partial, \lambda)v_{\Delta}^\beta, \quad \partial \epsilon_{\beta} = \beta \epsilon_{\beta} + a(\partial)v_{\Delta}^\beta,$$

(3.16)

where $f(\partial, \lambda)$, $g(\partial, \lambda) \in \mathbb{C} [\partial, \lambda]$ and $a(\partial) \in \mathbb{C} [\partial]$.

**Lemma 3.4:** All trivial extensions of the form (3.15) are given by (3.16), where $f(\partial, \lambda) = (\partial + \alpha + (\Delta + \frac{1}{2}) \lambda)h(\partial + \lambda)$, $g(\partial, \lambda) = (\partial + \alpha + 2 \Delta \lambda)h(\partial + \lambda)$, $a(\partial) = (\partial - \beta)h(\partial)$, and $h$ is a polynomial.

Applying (2.9) and (2.12) to the vector $\epsilon_{\beta}$ we get, respectively (after a shift by $\alpha$),

$$(\partial + (\Delta + \frac{1}{2}) \lambda)f(\partial + \lambda, \mu) - (\partial + (\Delta + \frac{1}{2}) \mu)f(\partial + \mu, \lambda) = (\lambda - \mu)f(\partial, \lambda + \mu),$$

(3.17)

$$g(\partial + \lambda, \mu) + g(\partial + \mu, \lambda) = 2f(\partial, \lambda + \mu).$$

(3.18)

Hence we may assume that $f$ is homogeneous in $\partial$ and $\lambda$.

Applying (2.10) and (2.13) to $\epsilon_{\beta}$, respectively, gives

$$(\partial + \lambda - (\alpha + \beta))f(\partial, \lambda) = a(\partial + \lambda)(\partial + (\Delta + \frac{1}{2}) \lambda),$$

(3.19)

$$(\partial + \lambda - (\alpha + \beta))g(\partial, \lambda) = a(\partial + \lambda)(\partial + 2 \Delta \lambda).$$

(3.20)

Set $\mu = 0$ in (3.17) we get

$$f(\partial + \lambda, 0)(\partial + (\Delta + \frac{1}{2}) \lambda) = (\partial + \lambda)f(\partial, \lambda),$$
and hence if \( f \) is not a constant, then
\[
f(\partial, \lambda) = (\partial + (\Delta + \frac{1}{2}) \lambda) h(\partial + \lambda),
\]
(3.21)

where \( h(\partial + \lambda) = [f(\partial + \lambda, 0)]/(\partial + \lambda) \). Substituting (3.21) into (3.19) we get, after simplification,
\[
a(\partial + \lambda) = (\partial + \lambda - (\alpha + \beta)) h(\partial + \lambda).
\]
(3.22)

We now combine (3.22) and (3.20) and find that
\[
g(\partial, \lambda) = (\partial + 2\Delta \lambda) h(\partial + \lambda).
\]
(3.23)

However, (3.21)–(3.23) is a trivial cocycle by Lemma 3.4.

Therefore, we may assume that \( f = 1 \). By (3.19) \( \alpha + \beta = 0 \), \( \Delta = \frac{1}{2} \), and \( a = 1 \). Furthermore, (3.18) implies that \( g = 1 \). This extension is nontrivial.

**Theorem 3.2:** Nontrivial extensions of the form (3.15) exist if and only if \( \alpha + \beta = 0 \) and \( \Delta = \frac{1}{2} \). In this case, the unique extension is given by (3.16), where
\[
f(\partial, \lambda) = g(\partial, \lambda) = a(\partial) = a_0, \quad a_0 \neq 0.
\]

**Remark 3.2:** The extension of Theorem 3.2 is given as follows: It is more convenient to reverse the parity and consider the \( E_c \) of codimension 1. We distinguish two cases, namely extensions between \( \mathcal{N}(\alpha, 0) \) and \( \mathcal{N}(\alpha, 0) \). Thus we have an exact sequence
\[
0 \rightarrow \mathcal{N}'(\alpha, 1) \rightarrow \mathcal{N}(\alpha, 0) \rightarrow \mathbb{C} \rightarrow 0.
\]

The corresponding conformal module can be realized as follows: Consider \( E^c = \mathbb{C}[t, t^{-1}, \xi] e^{-\alpha t dt} d\xi \) and the residue function \( \hat{\phi}: E^c \rightarrow \mathbb{C} \), which assigns to each element in \( E^c \) its coefficient of \( \tau^{-1} \xi dt d\xi \). It is readily checked that the kernel of \( \hat{\phi} \) is spanned by elements of the form \( t^m e^{-\alpha t dt} d\xi \) and \( (t^n - \alpha t^{n+1}/(n+1)) \xi e^{-\alpha t dt} d\xi \) with \( n, m \in \mathbb{Z} \) and \( n = -1 \), and is a submodule of codimension 1.

**IV. EXTENSIONS OF NON-ONE-DIMENSIONAL NEVEU–SCHWARZ MODULES**

In this section we classify extensions between two non-one-dimensional finite-rank irreducible modules over \( R(\mathfrak{g}) \). We distinguish two cases, namely extensions between \( \mathcal{N}(\alpha, \Delta') \) and \( \mathcal{N}'(\beta, \Delta) \), and between \( \mathcal{N}(\alpha, \Delta') \) and \( \mathcal{N}(\beta, \Delta) \).

We first consider the case of
\[
0 \rightarrow \mathcal{N}(\alpha, \Delta') \rightarrow E \rightarrow \mathcal{N}'(\beta, \Delta) \rightarrow 0.
\]
(4.1)

We write \( \mathcal{N}(\alpha, \Delta') = \mathbb{C}[\partial] v + \mathbb{C}[\partial] v^\xi \) and \( \mathcal{N}'(\beta, \Delta) = \mathbb{C}[\partial] w + \mathbb{C}[\partial] w^\xi \). Here, as usual, the superscript \( \xi \) is to denote that the vector under consideration is an odd vector. As a \( \mathbb{C}[\partial] \)-module, \( E \) is isomorphic to \( \mathcal{N}(\alpha, \Delta') \oplus \mathcal{N}'(\beta, \Delta) \), so that we may make this identification. \( \mathcal{N}(\alpha, \Delta') \) is an \( R(\mathfrak{g}) \)-submodule with action given by (2.15). \( R(\mathfrak{g}) \) acts on \( w \) and \( w^\xi \) as follows:
\[
L_\lambda w = (\partial + \beta + \Delta \lambda) w + f(\partial, \lambda) v,
\]
\[
L_\lambda w^\xi = (\partial + \beta + (\Delta - \frac{1}{2}) \lambda) w^\xi + g(\partial, \lambda) v^\xi,
\]
\[
G_\lambda w = (\partial + \beta + (2\Delta - 1) \lambda) w^\xi + R(\partial, \lambda) v^\xi,
\]
\[
G_\lambda w^\xi = w + S(\partial, \lambda) v^\xi,
\]
(4.2)
where \( f(\partial, \lambda) \), \( g(\partial, \lambda) \), \( R(\partial, \lambda) \), and \( S(\partial, \lambda) \) are polynomials in \( \partial \) and \( \lambda \).

Trivial cocycles are given as follows:

**Lemma 4.1:** All trivial extensions of the form (4.1) are given by (4.2), where

\[
\begin{align*}
  f(\partial, \lambda) &= (\partial + \alpha + \Delta \lambda) h(\partial + \lambda) - (\partial + \beta + \Delta \lambda) h(\partial), \\
  g(\partial, \lambda) &= (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) k(\partial + \lambda) - (\partial + \beta + (\Delta - \frac{1}{2}) \lambda) k(\partial), \\
  R(\partial, \lambda) &= h(\partial + \lambda) - (\partial + \beta + (2\Delta - 1) \lambda) k(\partial), \\
  S(\partial, \lambda) &= (\partial + \beta + 2\Delta') \lambda) k(\partial + \lambda) - h(\partial),
\end{align*}
\]

and \( h \) and \( k \) are polynomials.

Applying (2.9), (2.11), and (2.12) to \( w \) and \( w^\xi \) we obtain the following six identities:

\[
\begin{align*}
  (\lambda - \mu) f(\partial, \lambda + \mu) &= (\partial + \beta + \lambda + \Delta \mu) f(\partial, \lambda) + (\partial + \alpha + \Delta' \lambda) f(\partial + \mu, \lambda) \\
  &\quad - (\partial + \beta + \mu + \Delta \lambda) f(\partial, \mu) - (\partial + \alpha + \Delta' \mu) f(\partial + \mu, \lambda), \quad (4.3) \\
  (\lambda - \mu) g(\partial, \lambda + \mu) &= (\partial + \beta + \lambda + (\Delta - \frac{1}{2}) \mu) g(\partial, \lambda) + (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) g(\partial + \lambda, \mu) \\
  &\quad - (\partial + \beta + \mu + (\Delta - \frac{1}{2}) \lambda) g(\partial, \mu) - (\partial + \alpha + (\Delta' + \frac{1}{2}) \mu) g(\partial + \mu, \lambda), \quad (4.4) \\
  \left( \frac{\lambda}{2} - \mu \right) R(\partial, \lambda + \mu) &= (\partial + \beta + \lambda + (2\Delta - 1) \mu) g(\partial, \lambda) + (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) R(\partial + \mu, \lambda) \\
  &\quad - (\partial + \beta + \mu + \Delta \lambda) R(\partial, \mu) - f(\partial + \mu, \lambda), \quad (4.5) \\
  \left( \frac{\lambda}{2} - \mu \right) S(\partial, \lambda + \mu) &= f(\partial, \lambda) + (\partial + \alpha + \Delta' \lambda) S(\partial + \lambda, \mu) - (\partial + \beta + \mu + (\Delta - \frac{1}{2}) \lambda) S(\partial, \mu) \\
  &\quad - (\partial + \alpha + 2\Delta' \mu) g(\partial + \mu, \lambda), \quad (4.6) \\
  2 f(\partial, \lambda + \mu) &= (\partial + \beta + \lambda + (2\Delta - 1) \mu) S(\partial, \lambda) + (\partial + \alpha + 2\Delta' \lambda) R(\partial + \mu, \lambda) \\
  &\quad + (\partial + \beta + \mu + (2\Delta - 1) \lambda) S(\partial, \mu) + (\partial + \alpha + 2\Delta' \mu) R(\partial + \mu, \lambda), \quad (4.7) \\
  2 g(\partial, \lambda + \mu) &= R(\partial, \lambda) + S(\partial + \lambda, \mu) + R(\partial, \mu) + S(\partial + \mu, \lambda). \quad (4.8)
\end{align*}
\]

Now if \( \alpha \neq \beta \), Theorem 2.4, Lemma 4.1, (4.3), and (4.4) tell us that we may assume that \( f(\partial, \lambda) = g(\partial, \lambda) = 0 \). Setting \( \mu = 0 \) in (4.5) we then get

\[
\frac{\lambda}{2} R(\partial, \lambda) = (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) R(\partial + \lambda, 0) - (\partial + \beta + \Delta \lambda) R(\partial, 0). \quad (4.9)
\]

Putting \( \lambda = 0 \) in (4.9) we see that \( (\alpha - \beta) R(\partial, 0) = 0 \), which we may then plug in back into (4.9) to conclude that \( R(\partial, \lambda) = 0 \).

Now \( g(\partial, \lambda) = R(\partial, \lambda) = 0 \) in (4.8) gives (with \( \mu = 0 \))

\[
S(\partial + \lambda, 0) + S(\partial, \lambda) = 0. \quad (4.10)
\]

We set \( \lambda = 0 \) in (4.10) and conclude that \( S(\partial, 0) = 0 \), which implies that \( S(\partial, \lambda) = 0 \) due to (4.10).

Thus our discussion above shows that for our problem we may assume from now on that \( \alpha = \beta \). Replacing \( \partial + \alpha \) by \( \partial \) we may rewrite (4.3)–(4.8) in homogeneous form:
\[(\lambda - \mu) f(\partial + \mu, \lambda) = (\partial + \mu + \Delta \mu) f(\partial + \lambda, \mu) + (\partial + \Delta' \lambda) f(\partial + \mu, \lambda) - (\partial + \mu + \Delta \mu) f(\partial + \mu, \lambda), \quad (4.11)\]

\[(\lambda - \mu) g(\partial + \mu, \lambda) = (\partial + \mu + (\Delta - 1/2) \mu) g(\partial + \lambda, \mu) + (\partial + (\Delta' + 1/2) \lambda) g(\partial + \mu, \lambda) - (\partial + \mu + (\Delta - 1/2) \mu) g(\partial + \mu, \lambda), \quad (4.12)\]

\[\left(\frac{\lambda}{2} - \mu\right) R(\partial, \lambda + \mu) = (\partial + \mu + (2\Delta - 1) \mu) g(\partial, \mu) + (\partial + (\Delta' + 1/2) \lambda) R(\partial + \mu, \lambda) - (\partial + \mu + (\Delta - 1/2) \lambda) R(\partial + \mu, \lambda), \quad (4.13)\]

\[\left(\frac{\lambda}{2} - \mu\right) S(\partial, \lambda + \mu) = f(\partial, \mu) + (\partial + \Delta' \lambda) S(\partial + \mu, \mu) - (\partial + \mu + (\Delta - 1/2) \lambda) S(\partial, \mu) - (\partial + 2\Delta' \mu) g(\partial + \mu, \lambda), \quad (4.14)\]

\[2 f(\partial, \lambda + \mu) = (\partial + \lambda + (2\Delta - 1) \mu) S(\partial, \lambda) + (\partial + 2\Delta' \lambda) R(\partial + \mu, \lambda) + (\partial + \mu + (2\Delta - 1) \mu) S(\partial, \mu) + (\partial + 2\Delta' \mu) R(\partial + \mu, \lambda), \quad (4.15)\]

\[2 g(\partial, \lambda + \mu) = R(\partial, \lambda) + S(\partial + \mu, \mu) + R(\partial, \mu) + S(\partial + \mu, \lambda). \quad (4.16)\]

The first task now is to determine solutions to (4.11)–(4.16) in the case when \(f(\partial, \lambda) = g(\partial, \lambda) = 0\).

**Lemma 4.2:** Let \(F(\partial, \lambda)\) be a nonzero homogeneous polynomial of degree \(m\) satisfying

\[\left(\frac{\lambda}{2} - \mu\right) F(\partial, \lambda + \mu) = (\partial + (\Delta' + 1/2) \lambda) F(\partial + \lambda, \mu) - (\partial + \Delta \lambda + \mu) F(\partial, \mu), \quad (4.17)\]

where \(\Delta', \Delta \in \mathbb{C}\). Then \(\Delta - \Delta' = m\), \(m \geq 2\) and all the solutions (up to a scalar) are given by

(i) \(m = 0\) and \(F(\partial, \lambda) = 1\);
(ii) \(m = 1\) and \(F(\partial, \lambda) = \partial + (2\Delta' + 1) \lambda\);
(iii) \(m = 2, \Delta = \frac{3}{2}, \Delta' = \frac{1}{2}\), and \(F(\partial, \lambda) = \partial^2 + 2\partial \lambda\); and
(iv) \(m = 2, \Delta = 1, \Delta' = -1\), and \(F(\partial, \lambda) = \partial^2 - \lambda^2\).

**Proof:** Substituting \(F(\partial, \lambda) = \sum_{i=0}^{m} a_i \partial^{m-i} \lambda^i\) and \(\mu = 0\) into (4.17) we obtain

\[\left(\frac{\lambda}{2}\right) \sum_{i=0}^{m} a_i \partial^{m-i} \lambda^i = (\partial + (\Delta' + 1/2) \lambda) a_0 (\partial + \lambda) + (\partial + \Delta \lambda + \mu) a_0 \partial^m. \quad (4.18)\]

Taking the coefficient of \(\partial^m \lambda\) we see that \(\frac{1}{2} a_0 = a_0 (m + \Delta' - \Delta + \frac{1}{2})\). Hence if \(a_0 \neq 0\), we must have \(m = \Delta - \Delta'\).

On the other hand, if \(a_0 = 0\), we have \(F(\partial, \lambda) = 0\) by (4.18), and so we may assume that \(a_0 \neq 0\) and \(m = \Delta - \Delta'\).

Solving (4.18) for other coefficients give

\[\frac{1}{2} a_i = a_0 \left( m \atop i + 1 \right) + a_0 (\Delta' + \frac{1}{2}) \left( m \atop i \right), \quad 1 \leq i \leq m. \quad (4.19)\]

Substituting \(F(\partial, \lambda) = \sum_{i=0}^{m} a_i \partial^{m-i} \lambda^i\) into (4.17) we have
\[\left(\frac{\lambda}{2} - \mu\right) \sum_{i=0}^{m} a_i \partial^{m-i}(\lambda + \mu)^i = (\partial + (\Delta' + \frac{i}{2})\lambda) \sum_{i=0}^{m} a_i (\partial + \lambda)^{m-i}\lambda^{-i} - (\partial + \Delta\lambda + \mu) \sum_{i=0}^{m} a_i \partial^{m-i}(\lambda + \mu)^i. \tag{4.20}\]

The coefficient of \(\lambda^m\mu^0\) in (4.20) with \(m \geq 2\) gives \((m/2)a_m - a_m = (\Delta' + \frac{i}{2})a_1\), and hence by (4.19) \(a_0(m/2 - 1)(\Delta' + \frac{i}{2}) = (\Delta' + \frac{i}{2})(2a_0(\frac{i}{2}) + 2a_0(\Delta' + \frac{i}{2})m)\), which in the case when \(\Delta' + \frac{i}{2} \neq 0\) gives

\[m^2 + (2\Delta' - \frac{i}{2})m + 1 = 0. \tag{4.21}\]

Similarly collecting the coefficient of \(\lambda^2\mu^{m-1}\) with \(m \geq 2\) gives [using (4.19) again]

\[m^2 + (2\Delta' - 1)m + 2 = 0. \tag{4.22}\]

Therefore, it follows from (4.21) and (4.22) that \(m = 2\). Hence we have proved that if \(\Delta' + \frac{i}{2} \neq 0\) and \(m \geq 2\), then \(m = 2\).

Suppose now that \(\Delta' + \frac{i}{2} = 0\) and \(m \geq 3\). In this case the coefficient of \(\partial\lambda^{m-1}\mu\) in (4.20) gives \([a_{m-1}(m-1)/2] - a_{m-1} = a_1\), from which we conclude, using (4.19), that

\[a_0(m - 1) - 2a_0 = a_0m(m - 1).\]

Since \(a_0 \neq 0\), we have then \(m^2 - 2m + 3 = 0\), which certainly cannot happen, since \(m\) would not be an integer.

Thus we have only three possible cases for \(m\), namely, \(m = 0\), \(1\), \(2\).

Suppose that \(m = 0\). Then \(\Delta = \Delta'\) and \(F(\partial, \lambda) = \text{constant}\) is a solution to (4.17).

Next suppose that \(m = 1\). Then again it is easily checked that \(F(\partial, \lambda) = \partial + (2\Delta' + 1)\lambda\) is a unique, up to a constant, solution to (4.17).

Finally, consider the case \(m = 2\). By (4.19) we may assume that

\[F(\partial, \lambda) = \partial^2 + (4 + 4\Delta')\partial\lambda + (2\Delta' + 1)\lambda^2.\]

Plugging this back into (4.17) we obtain

\[
\left(\frac{\lambda}{2} - \mu\right)\left(\partial^2 + (4\Delta' + 4)\partial\lambda + (2\Delta' + 1)(\lambda + \mu)^2\right)
= (\partial + (\Delta' + \frac{i}{2})\lambda)((\partial + \lambda)^2 + (4\Delta' + 4)\partial\lambda + (2\Delta' + 1)\mu^2)
- (\partial + \Delta\lambda + \mu)(\partial^2 + (4\Delta' + 4)\partial\mu + (2\Delta' + 1)\mu^2). \tag{4.23}\]

Collecting the coefficient of \(\lambda^2\mu\) in (4.23) gives, after simplification,

\[(2\Delta' + 1)(2\Delta' + 2) = 0,
\]

which implies that \(\Delta' = -1\) or \(\Delta' = -\frac{i}{2}\).

**Proposition 4.1:** Let \(f(\partial, \lambda), g(\partial, \lambda), R(\partial, \lambda),\) and \(S(\partial, \lambda)\) satisfy (4.11)–(4.16). Suppose that \(f(\partial, \lambda) = g(\partial, \lambda) = 0\). Then the extension of the form (4.1) associated to these polynomials is a trivial extension.

**Proof:** Substituting \(f(\partial, \lambda) = 0\) and \(g(\partial, \lambda) = 0\) in (4.13) and (4.14) we see that

\[
\left(\frac{\lambda}{2} - \mu\right) R(\partial, \lambda + \mu) = (\partial + (\Delta' + \frac{i}{2})\lambda) R(\partial, \lambda + \mu) - (\partial + \mu + \Delta\lambda) R(\partial, \mu),
\]

\[
\left(\frac{\lambda}{2} - \mu\right) S(\partial, \lambda + \mu) = (\partial + \Delta'\lambda) S(\partial, \lambda + \mu) - (\partial + \mu + (\Delta - \frac{i}{2})\lambda) S(\partial, \mu).
\]
Now if \( R(\bar{\vartheta},\lambda) = 0 \), it follows easily from (4.16) that \( S(\bar{\vartheta},\lambda) = 0 \), and vice versa. Therefore, we may assume that both \( R(\bar{\vartheta},\lambda) \) and \( S(\bar{\vartheta},\lambda) \) are nonzero, and hence by (4.16) \( R(\bar{\vartheta},\lambda) \) and \( S(\bar{\vartheta},\lambda) \) are homogeneous of the same degree. However, Lemma 4.2 only allows three possible cases.

In the first case we have \( \Delta = \Delta' \) with \( R(\bar{\vartheta},\lambda) = c \) and \( S(\bar{\vartheta},\lambda) = d \), where \( c,d \in \mathbb{C} \). By (4.16) we have \( c = -d \). This, however, is a trivial extension by Lemma 4.1 \((h = c, k = 0)\).

In the second case we have \( \Delta - \Delta' = 1 \) and \( R(\bar{\vartheta},\lambda) = c(\bar{\vartheta} + (2\Delta' + 1)\lambda) \) and \( S(\bar{\vartheta},\lambda) = d(\bar{\vartheta} + 2\Delta' \lambda) \), \( c,d \in \mathbb{C} \). Again (4.16) tells us that \( c = -d \). However, this is also a trivial extension by Lemma 4.1 \((h = 0, k = -c \bar{\vartheta})\).

Suppose now that \((f(\bar{\vartheta},\lambda), g(\bar{\vartheta},\lambda)) \neq (0,0)\). We may assume that \( f(\bar{\vartheta},\lambda) \), \( g(\bar{\vartheta},\lambda) \), \( R(\bar{\vartheta},\lambda) \), and \( S(\bar{\vartheta},\lambda) \) are homogeneous polynomials of degree \( m + 1 \), \( m \), \( m \), and \( m \), respectively, where \( m > 0 \). [The case \( m = -1 \) is ruled out by (4.15).] We write

\[
R(\bar{\vartheta},\lambda) = \sum_{i=0}^{m} a_i \bar{\vartheta}^{m-i} \lambda^i, \quad S(\bar{\vartheta},\lambda) = \sum_{i=0}^{m} b_i \bar{\vartheta}^{m-i} \lambda^i.
\]

We set \( \mu = 0 \) in (4.13) and (4.14) and obtain

\[
\frac{\lambda}{2} R(\bar{\vartheta},\lambda) = (\bar{\vartheta} + \lambda) g(\bar{\vartheta},\lambda) + (\bar{\vartheta} + (\Delta' + \frac{1}{2})\lambda) a_0 (\bar{\vartheta} + \lambda)^m - (\bar{\vartheta} + \Delta \lambda) a_0 \bar{\vartheta}^m - f(\bar{\vartheta},\lambda), \quad (4.24)
\]

\[
\frac{\lambda}{2} S(\bar{\vartheta},\lambda) = f(\bar{\vartheta},\lambda) - \bar{\vartheta} g(\bar{\vartheta},\lambda) + b_0 (\bar{\vartheta} + \Delta' \lambda) (\bar{\vartheta} + \lambda)^m - b_0 (\bar{\vartheta} + (\Delta - \frac{1}{2}) \lambda) \bar{\vartheta}^m. \quad (4.25)
\]

Note that (4.24) and (4.25) imply that \( R(\bar{\vartheta},\lambda) \) and \( S(\bar{\vartheta},\lambda) \) are uniquely determined by the pair \((f(\bar{\vartheta},\lambda), g(\bar{\vartheta},\lambda))\). Since \((f(\bar{\vartheta},\lambda), g(\bar{\vartheta},\lambda))\) is given by polynomials in the list of Theorem 2.4 with \( \deg f(\bar{\vartheta},\lambda) = \deg g(\bar{\vartheta},\lambda) + 1 \) due to Lemma 4.1, (4.11), and (4.12), we may use (4.24) and (4.25) to define \( R(\bar{\vartheta},\lambda) \) and \( S(\bar{\vartheta},\lambda) \), and then use Eqs. (4.13)–(4.16) to check for consistency. This computation was carried out by Maple V and below we will first give all such pairs \((f(\bar{\vartheta},\lambda), g(\bar{\vartheta},\lambda))\) and then give the solutions.

All possible pairs of \((f(\bar{\vartheta},\lambda), g(\bar{\vartheta},\lambda))\) are given (up to a constant factor) as follows (see Theorem 2.4 for notation):

1. \( m = 7 \), \( \Delta' + \frac{1}{2} = (5 \pm \sqrt{19})/2 \), \( \Delta = (7 \pm \sqrt{19})/2 \), \( f(\bar{\vartheta},\lambda) = 0 \) and \( g(\bar{\vartheta},\lambda) = f_2(\bar{\vartheta},\lambda) \) with \( \Delta' \) replaced by \( \Delta' + \frac{1}{2} \).
2. \( m = 6 \).
   - \( \Delta' = (5 \pm \sqrt{19})/2 \), \( \Delta = (7 \pm \sqrt{19})/2 \), \( f(\bar{\vartheta},\lambda) = f_2(\bar{\vartheta},\lambda) \), and \( g(\bar{\vartheta},\lambda) = 0 \).
   - \( \Delta - \frac{1}{2} = 5 \), \( \Delta' + \frac{1}{2} = 0 \), \( f(\bar{\vartheta},\lambda) = 0 \), and \( g(\bar{\vartheta},\lambda) = f_4(\bar{\vartheta},\lambda) \).
   - \( \Delta - \frac{1}{2} = 1 \), \( \Delta' + \frac{1}{2} = -4 \), \( f(\bar{\vartheta},\lambda) = 0 \), and \( g(\bar{\vartheta},\lambda) = f_6(\bar{\vartheta},\lambda) \).
3. \( m = 5 \) and \( \Delta - \Delta' = 5 \).
   - \( f(\bar{\vartheta},\lambda) = 0 \) and \( g(\bar{\vartheta},\lambda) = f_5(\bar{\vartheta},\lambda) \) with \( \Delta' \) replaced by \( \Delta' + \frac{1}{2} \).
   - \( \Delta = 5 \), \( f(\bar{\vartheta},\lambda) = f_5(\bar{\vartheta},\lambda) \), \( g(\bar{\vartheta},\lambda) = c f_5(\bar{\vartheta},\lambda) \) with \( \Delta' \) replaced by \( \Delta' + \frac{1}{2} \), \( c \in \mathbb{C} \).
   - \( \Delta = 1 \), \( f(\bar{\vartheta},\lambda) = f_6(\bar{\vartheta},\lambda) \), \( g(\bar{\vartheta},\lambda) = c f_5(\bar{\vartheta},\lambda) \) with \( \Delta' \) replaced by \( \Delta' + \frac{1}{2} \), \( c \in \mathbb{C} \).
4. \( m = 4 \), \( \Delta - \Delta' = 4 \), \( f(\bar{\vartheta},\lambda) = c_1 f_4(\bar{\vartheta},\lambda) \), and \( g(\bar{\vartheta},\lambda) = c_2 f_4(\bar{\vartheta},\lambda) \), \( c_1, c_2 \in \mathbb{C} \) with \((c_1, c_2) \neq (0,0)\).
5. \( m = 3 \), \( \Delta - \Delta' = 3 \), \( f(\bar{\vartheta},\lambda) = c_1 f_4(\bar{\vartheta},\lambda) \), and \( g(\bar{\vartheta},\lambda) = c_2 f_4(\bar{\vartheta},\lambda) \), \( c_1, c_2 \in \mathbb{C} \) with \((c_1, c_2) \neq (0,0)\).
6. \( m = 2 \), \( \Delta - \Delta' = 2 \).
   - \( f(\bar{\vartheta},\lambda) = f_3(\bar{\vartheta},\lambda) \) and \( g(\bar{\vartheta},\lambda) = 0 \).
(ii) \( \Delta - \frac{1}{2} = 1, \Delta' + \frac{1}{2} = 0, f(\partial, \lambda) = c_1 f_3(\partial, \lambda), \) and \( g(\partial, \lambda) = c_2 \partial \lambda + c_3 \lambda^2 \) with \( c_1, c_2, c_3 \in \mathbb{C} \) and \((c_2, c_3) \neq (0, 0)\).

(7) \( m = 1. \)

(i) \( \Delta = 1, \Delta' = 0, f(\partial, \lambda) = c_1 \partial \lambda + c_2 \lambda^2, g(\partial, \lambda) = c_3 \lambda, \) where \( c_1, c_2, c_3 \in \mathbb{C} \) with \((c_1, c_2) \neq (0, 0)\).

(ii) \( \Delta - \frac{1}{2} = \Delta' + \frac{1}{2} = 0, f(\partial, \lambda) = 0, g(\partial, \lambda) = \lambda. \)

(iii) \( \Delta - \frac{1}{2} = 1, \Delta' + \frac{1}{2} = 0, f(\partial, \lambda) = 0, g(\partial, \lambda) = \partial. \)

(8) \( m = 0. \)

(i) \( \Delta = \Delta', f(\partial, \lambda) = \lambda, g(\partial, \lambda) = 0. \)

(ii) \( \Delta = 1, \Delta' = 0, f(\partial, \lambda) = \partial, g(\partial, \lambda) = c, \) where \( c \in \mathbb{C}. \)

The solutions to (4.11)-(4.16) are as follows:

There are no solutions for cases 1–4, 7(ii), 7(iii), and 8(i).

(5) \( c_1 = 3c_2 \) and the unique (up to a scalar) solution is given by

\[
\begin{align*}
    f(\partial, \lambda) &= 3 \partial \lambda^2 (\partial + \lambda), \\
    R(\partial, \lambda) &= 2 \lambda (\lambda^2 - \partial^2), \\
    g(\partial, \lambda) &= \lambda^2 (2 \partial + \lambda), \\
    S(\partial, \lambda) &= 2 \partial \lambda (\partial + 2 \lambda).
\end{align*}
\]

(6i) \( \Delta' \neq - \frac{1}{2} \) and the unique (up to a scalar) solution is

\[
\begin{align*}
    f(\partial, \lambda) &= \lambda^2 (2 \partial + \lambda), \\
    R(\partial, \lambda) &= \frac{2}{2 \Delta' + 1} (\partial^2 + 2 \partial \lambda), \\
    g(\partial, \lambda) &= 0, \\
    S(\partial, \lambda) &= \frac{2}{2 \Delta' + 1} (\lambda^2 - \partial^2).
\end{align*}
\]

(6ii) \( c_1 = c_3, c_2 = 0, \) and the solutions are (up to a scalar) given by

\[
\begin{align*}
    f(\partial, \lambda) &= \lambda^2 (2 \partial + \lambda), \\
    R(\partial, \lambda) &= c (\partial^2 + 2 \partial \lambda) - 2 \partial \lambda, \\
    g(\partial, \lambda) &= \lambda^2, \\
    S(\partial, \lambda) &= c (\lambda^2 - \partial^2) + 2 (\partial \lambda + \lambda^2).
\end{align*}
\]

(7i) \( c_1 = c_3 \) and the solutions are

\[
\begin{align*}
    f(\partial, \lambda) &= c_1 \partial \lambda + c_2 \lambda^2, \\
    R(\partial, \lambda) &= c_3 (\partial + \lambda) + c_1 (\partial - \lambda) + 2 (c_1 - c_2) \lambda, \\
    g(\partial, \lambda) &= c_1 \lambda, \\
    S(\partial, \lambda) &= - c_3 \partial + 2 c_2 \lambda, \\
    c_1, c_2, c_3 \in \mathbb{C}.
\end{align*}
\]

(8ii) \( c = 1 \) and the unique (up to a scalar) solution is

\[
\begin{align*}
    f(\partial, \lambda) &= \partial, \\
    R(\partial, \lambda) &= 1, \\
    g(\partial, \lambda) &= 1, \\
    S(\partial, \lambda) &= 0.
\end{align*}
\]

We summarize the above discussion in the following theorem.

**Theorem 4.1:** Nontrivial extensions of the form (4.1) exist only if \( \alpha = \beta \) and \( \Delta - \Delta' = 1, 2, 3. \) The following is a complete list of values of \( \Delta \) and \( \Delta' \) along with the quadruple of polynomials \( f(\partial, \lambda), g(\partial, \lambda), R(\partial, \lambda), \) and \( S(\partial, \lambda) \) whose nonzero scalar multiples give rise to nontrivial extensions \( (\partial = \partial + \alpha): \)

(i) \( \Delta = 1, \Delta' = 0, \) and
\[ f(\vec{\alpha}, \lambda) = c_1 \vec{\alpha} + c_2 \lambda^2 + c_3 \vec{\alpha}, \quad g(\vec{\alpha}, \lambda) = c_1 \lambda + c_3, \]

\[ R(\vec{\alpha}, \lambda) = c_4 (\vec{\alpha} + \lambda) + 2(c_1 - c_2) \lambda + c_3, \quad S(\vec{\alpha}, \lambda) = -c_4 \vec{\alpha} + 2c_2 \lambda, \]

where \( c_1, c_2, c_3, c_4 \in \mathbb{C} \), and \((c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)\).

(ii) \( \Delta = \frac{1}{2} \), \( \Delta' = -\frac{1}{2} \), and

\[ f(\vec{\alpha}, \lambda) = \lambda^2 (2 \vec{\alpha} + \lambda), \quad g(\vec{\alpha}, \lambda) = \lambda^2, \]

\[ R(\vec{\alpha}, \lambda) = c(2\vec{\alpha} + 2\lambda) - 2\vec{\alpha}, \quad S(\vec{\alpha}, \lambda) = c(\lambda^2 - \vec{\alpha}^2) + 2(\vec{\alpha} \lambda + \lambda^2), \quad c \in \mathbb{C}. \]

(ii') \( \Delta - \Delta' = 2, \Delta' \neq \frac{1}{2}, \) and

\[ f(\vec{\alpha}, \lambda) = \lambda^2 (2 \vec{\alpha} + \lambda), \quad g(\vec{\alpha}, \lambda) = 0, \]

\[ R(\vec{\alpha}, \lambda) = \frac{2}{2\Delta' + 1}(2\vec{\alpha} + 2\lambda), \quad S(\vec{\alpha}, \lambda) = \frac{2}{2\Delta' + 1} (\lambda^2 - \vec{\alpha}^2). \]

(iii) \( \Delta - \Delta' = 3, \)

\[ f(\vec{\alpha}, \lambda) = 3 \vec{\alpha} \lambda^2 (\vec{\alpha} + \lambda), \quad g(\vec{\alpha}, \lambda) = \lambda^2 (2 \vec{\alpha} + \lambda), \]

\[ R(\vec{\alpha}, \lambda) = 2\lambda (\lambda^2 - \vec{\alpha}^2), \quad S(\vec{\alpha}, \lambda) = 2\vec{\alpha} \lambda (\vec{\alpha} + 2\lambda). \]

The space of extensions is four-dimensional in the case (i), two-dimensional in the case (ii), one-dimensional in the case (ii') and (iii), and trivial for all other values of \( \Delta \) and \( \Delta' \).

Below we provide a formula to translate the extensions obtained above into the language of conformal modules. The conformal module of \( E \) will be denoted by \( E' \). As a vector space \( E' = \mathbb{C} \[ t, t^{-1}, \xi \] e^{-at} \omega^{1/2 - \Delta'} \oplus \mathbb{C} \[ t, t^{-1}, \xi \] e^{-at} \omega^{1 - \Delta} \), where the parity is given by \( p(\omega^{1/2 - \Delta'}) = 1 \) and \( p(\omega^{1 - \Delta}) = 0 \). On the space \( \mathbb{C} \[ t, t^{-1}, \xi \] e^{-at} \omega^{1/2 - \Delta'} \) it acts as in Example 2.2, while on the space \( \mathbb{C} \[ t, t^{-1}, \xi \] e^{-at} \omega^{1 - \Delta} \) the action is as follows \( \{ f(t) \in \mathbb{C} \[ t, t^{-1}, \xi \]| g(t) \in \mathbb{C} \[ t, t^{-1}, \xi \] \} : \)

\[ D^{(\ell)} \xi \omega^{1-\Delta} = (f(t)g'(t)+(1-\Delta)f'(t)g(t))\omega^{1-\Delta} + \sum_{i,k} a_{ik} (-1)^i \]

\[ \times \left( \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \\ \end{array} \right) f^{(k+i-j)}(t)g^{(j)}(t) \right) \xi \omega^{(1/2)-\Delta'}, \]

\[ D^{(\ell)} \xi \omega^{1-\Delta} = (f(t)g'(t)+(1-\Delta)f'(t)g(t))\xi \omega^{1-\Delta} + \sum_{i,k} b_{ik} (-1)^i \]

\[ \times \left( \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \\ \end{array} \right) f^{(k+i-j)}(t)g^{(j)}(t) \right) \omega^{(1/2)-\Delta'}, \]

\[ D^{(\ell)} \xi \omega^{1-\Delta} = (f(t)g'(t)+(2-\Delta)f'(t)g(t))\xi \omega^{1-\Delta} + \sum_{i,k} c_{ik} (-1)^i \]

\[ \times \left( \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \\ \end{array} \right) f^{(k+i-j)}(t)g^{(j)}(t) \right) \omega^{(1/2)-\Delta'}, \]
\[ D^{\xi(t)} g(t) \xi \omega^{1-\Delta} = -f(t) g(t) \omega^{1-\Delta} + \sum_{i,k} d_{ik} (-1)^i \left( \sum_{j=0}^i \left( \frac{i}{j} \right) f^{(k+i-j)}(t) g^{(j)}(t) \right) \xi \omega^{1/2-\Delta}, \]

where \( f(\vartheta, \lambda) = \sum_{i,k} a_{ik} \vartheta^k \lambda^i \), \( g(\vartheta, \lambda) = \sum_{i,k} b_{ik} \vartheta^k \lambda^i \), \( R(\vartheta, \lambda) = \sum_{i,k} c_{ik} \vartheta^k \lambda^i \), and \( S(\vartheta, \lambda) = \sum_{i,k} d_{ik} \vartheta^k \lambda^i \). Here \( h^{(i)}(t) \) denotes the \( i \)th derivative of the polynomial \( h(t) \) with respect to \( t \). The proof of these formulas are exactly the same as the proof for formula (3.13) in Ref. 2 and hence will be omitted.

Finally, consider extensions of the form
\[ 0 \rightarrow N(\alpha, \Delta') \rightarrow E \rightarrow N(\beta, \Delta) \rightarrow 0. \]  
(4.26)

We have \( E = N(\alpha, \Delta') \oplus N(\beta, \Delta) \) as \( \mathbb{C} \{ \partial \} \)-modules and we write \( N(\alpha, \Delta') = \mathbb{C} \{ \partial \} v + \mathbb{C} \{ \partial \} v^\xi \) and \( N(\beta, \Delta) = \mathbb{C} \{ \partial \} w + \mathbb{C} \{ \partial \} w^\xi \). The action of \( R(\vartheta) \) on the vectors \( v \) and \( v^\xi \) is given by (2.15), while on the vectors \( w \) and \( w^\xi \) it is given by

\[ L_\lambda w = (\partial + \beta + \Delta \lambda) w + f(\vartheta, \lambda) v, \]
\[ L_\lambda w^\xi = (\partial + \beta + (\Delta + \frac{1}{2}) \lambda) w^\xi + g(\vartheta, \lambda) v^\xi, \]
\[ G_\lambda w = w^\xi + R(\vartheta, \lambda) v^\xi, \]
\[ G_\lambda w^\xi = (\partial + \beta + 2 \Delta \lambda) w + S(\vartheta, \lambda) v, \]

where \( f(\vartheta, \lambda) \), \( g(\vartheta, \lambda) \), \( R(\vartheta, \lambda) \), and \( S(\vartheta, \lambda) \) are polynomials in \( \vartheta \) and \( \lambda \).

**Lemma 4.3:** All trivial extensions of the form (4.26) are given by (4.27), where

\[ f(\vartheta, \lambda) = (\partial + \alpha + \Delta' \lambda) h(\vartheta + \lambda) - (\partial + \beta + \Delta \lambda) h(\vartheta), \]
\[ g(\vartheta, \lambda) = (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) k(\vartheta + \lambda) - (\partial + \beta + (\Delta + \frac{1}{2}) \lambda) k(\vartheta), \]
\[ R(\vartheta, \lambda) = h(\vartheta + \lambda) - k(\vartheta), \]
\[ S(\vartheta, \lambda) = (\partial + \alpha + 2 \Delta' \lambda) k(\vartheta + \lambda) - (\partial + \beta + 2 \Delta \lambda) h(\vartheta), \]

and \( h \) and \( k \) are polynomials.

Applying (2.9), (2.11), and (2.12) to \( w \) and \( w^\xi \) we obtain

\[ (\lambda - \mu) f(\vartheta, \lambda + \mu) = (\partial + \beta + \lambda + \Delta \mu) f(\vartheta, \lambda) + (\partial + \alpha + \Delta' \lambda) f(\vartheta + \mu, \lambda), \]
\[ - (\partial + \beta + \mu + \Delta \lambda) f(\vartheta, \mu) - (\partial + \alpha + \Delta' \lambda) f(\vartheta + \mu, \lambda), \]
\[ (\lambda - \mu) g(\vartheta, \lambda + \mu) = (\partial + \beta + \lambda + (\Delta + \frac{1}{2}) \mu) g(\vartheta, \lambda) + (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) g(\vartheta + \lambda, \mu), \]
\[ - (\partial + \beta + \mu + (\Delta + \frac{1}{2}) \lambda) g(\vartheta, \mu) - (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) g(\vartheta + \mu, \lambda), \]
\[ \left( \frac{\lambda}{2} - \mu \right) R(\vartheta, \lambda + \mu) = g(\vartheta, \lambda) - f(\vartheta + \mu, \lambda) + (\partial + \alpha + (\Delta' + \frac{1}{2}) \lambda) R(\vartheta + \lambda, \mu), \]
\[ - (\partial + \beta + \mu + \Delta \lambda) R(\vartheta, \mu), \]
\[ \left( \frac{\lambda}{2} - \mu \right) S(\vartheta, \lambda + \mu) = (\partial + \alpha + \Delta' \lambda) S(\vartheta + \lambda, \mu) - (\partial + \beta + \mu + (\Delta + \frac{1}{2}) \lambda) S(\vartheta, \mu) \]
\[ + (\partial + \beta + \mu + 2 \Delta \mu) f(\vartheta, \lambda) - (\partial + \alpha + 2 \Delta' \mu) g(\vartheta + \mu, \lambda), \]
\[2f(\partial, \lambda + \mu) = S(\partial, \lambda) + (\partial + \alpha + 2\Delta' \lambda)S(\partial + \lambda, \mu) + S(\partial, \mu) + (\partial + \alpha + 2\Delta' \mu)S(\partial + \mu, \lambda)\]

\[2g(\partial, \lambda + \mu) = S(\partial + \lambda, \mu) + (\partial + \beta + \lambda + 2\Delta \mu)S(\partial, \lambda) + S(\partial + \mu, \lambda) + (\partial + \beta + \mu + 2\Delta \lambda)S(\partial + \mu, \lambda)\] (4.32)

Now if \(\alpha \neq \beta\), we may assume by Theorem 2.4, Lemma 4.3, (4.28), and (4.29) that \(f(\partial, \lambda) = g(\partial, \lambda) = 0\). We put \(\mu = 0\) in (4.30) and conclude that

\[\frac{\lambda}{2}R(\partial, \lambda) = (\partial + \alpha + (\Delta' + \frac{1}{2})\lambda)R(\partial + \lambda, 0) - (\partial + \beta + \Delta \lambda)R(\partial, 0).\] (4.34)

Now putting \(\lambda = 0\) in (4.34) gives \((\alpha - \beta)R(\partial, 0) = 0\), and hence \(R(\partial, 0) = 0\), from which it follows from (4.34) again that \(R(\partial, \lambda) = 0\).

With \(f(\partial, \lambda) = g(\partial, \lambda) = R(\partial, \lambda) = 0\) we have by (4.32)

\[S(\partial, \lambda) + S(\partial, \mu) = 0,\]

which gives \(S(\partial, \lambda) = 0\).

Thus we may assume from now on that \(\alpha = \beta\). Using a shift by \(\alpha\) as before, we may rewrite (4.28)–(4.33) in homogeneous form:

\[(\lambda - \mu)f(\partial, \lambda + \mu) = (\partial + \lambda + \Delta \mu)f(\partial, \lambda) + (\partial + \Delta' \lambda)f(\partial + \lambda, \mu)
- (\partial + \mu + \Delta \lambda)f(\partial, \mu) - (\partial + \Delta' \mu)f(\partial + \mu, \lambda),\] (4.35)

\[\frac{\lambda}{2}R(\partial, \lambda + \mu) = g(\partial, \lambda) - f(\partial + \mu, \lambda) + (\partial + (\Delta' + \frac{1}{2})\lambda)R(\partial + \lambda, \mu)
- (\partial + \mu + \Delta \lambda)R(\partial, \mu),\] (4.37)

\[\frac{\lambda}{2}S(\partial, \lambda + \mu) = (\partial + \Delta' \lambda)S(\partial, \lambda + \mu) - (\partial + \mu + (\Delta' + \frac{1}{2})\lambda)S(\partial, \mu)
+ (\partial + \mu + 2\Delta \mu)f(\partial, \lambda) - (\partial + 2\Delta' \mu)g(\partial + \mu, \lambda),\] (4.38)

\[2f(\partial, \lambda + \mu) = S(\partial, \lambda) + (\partial + 2\Delta' \lambda)R(\partial + \lambda, \mu) + S(\partial + \mu, \lambda) + (\partial + 2\Delta' \mu)R(\partial + \mu, \lambda),\] (4.39)

\[2g(\partial, \lambda + \mu) = S(\partial + \lambda, \mu) + (\partial + \lambda + 2\Delta \mu)R(\partial + \mu, \lambda) + S(\partial + \mu, \lambda) + (\partial + \mu + 2\Delta \lambda)R(\partial + \mu, \lambda),\] (4.40)

**Proposition 4.2:** Let \(f(\partial, \lambda), \ g(\partial, \lambda), \ S(\partial, \lambda), \) and \(R(\partial, \lambda)\) satisfy (4.35)–(4.40) with \(f(\partial, \lambda) = g(\partial, \lambda) = 0\). Then an extension of the form (4.26) associated to such a quadruple of polynomials is a trivial extension.

**Proof:** Substituting \(f(\partial, \lambda) = g(\partial, \lambda) = 0\) into (4.37) and (4.38) we get respectively

\[\frac{\lambda}{2}R(\partial, \lambda + \mu) = (\partial + (\Delta' + \frac{1}{2})\lambda)R(\partial + \lambda, \mu) - (\partial + \mu + \Delta \lambda)R(\partial, \mu),\]
Thus it is easy to see in our situation from (4.40) that \( R(\partial, \lambda) = 0 \) if and only if \( S(\partial, \lambda) = 0 \). Hence we may assume that \( R(\partial, \lambda) \neq 0 \) and \( S(\partial, \lambda) \neq 0 \). Thus we have \( \deg S(\partial, \lambda) = \deg R(\partial, \lambda) + 1 \). However, then Lemma 4.2 only leaves three possibilities.

In the first case we have \( \Delta = \Delta ', R(\partial, \lambda) = -c, \) and \( S(\partial, \lambda) = c(\partial + 2\Delta ') \), where \( c \in \mathbb{C} \). However, this corresponds to the trivial extension by Lemma 4.3 with \( h = 0 \) and \( k = c \).

In the second case we have \( \Delta = 1 \) and \( \Delta ' = 0 \) with \( R(\partial, \lambda) = c(\partial + \lambda) \) and \( S(\partial, \lambda) = -c(\partial^2 + 2\partial \lambda), c \in \mathbb{C} \). By Lemma 4.3 this is a trivial extension \( (h = c, k = 0) \).

Finally, in the last case we have \( \Delta = \frac{1}{2} \) and \( \Delta ' = -\frac{1}{2} \) with \( R(\partial, \lambda) = c \partial \) and \( S(\partial, \lambda) = -c(\partial^2 - \lambda^2) \). By Lemma 4.3 again this is a trivial extension \( (h = 0, k = -c \partial) \).

Equations (4.35)\textendash (4.40) allow us to assume that \( f(\partial, \lambda), g(\partial, \lambda), S(\partial, \lambda), \) and \( R(\partial, \lambda) \) are homogeneous polynomials in \( \partial \) and \( \lambda \) of degree \( m, m, m, \) and \( m - 1 \), respectively. We set \( R(\partial, \lambda) = \sum_{i=0}^{m-1} a_i \partial^{m-1-i} \lambda^i \) and \( S(\partial, \lambda) = \sum_{i=0}^{m} b_i \partial^m \lambda^{m-1} \). Letting \( \mu = 0 \) in (4.37) and (4.38) we obtain respectively

\[
\frac{\lambda}{2} R(\partial, \lambda) = g(\partial, \lambda) - f(\partial, \lambda) + a_0(\partial + (\Delta' + \frac{1}{2}) \lambda)(\partial + \lambda)^m - a_0 (\partial + \Delta) \partial^{m-1}, \tag{4.41}
\]

\[
\frac{\lambda}{2} S(\partial, \lambda) = b_0(\partial + \Delta') (\partial + \lambda)^m - b_0 (\partial + (\Delta' + \frac{1}{2}) \lambda) \partial^m + (\partial + \lambda) f(\partial, \lambda) - \partial g(\partial, \lambda). \tag{4.42}
\]

Thus \( R(\partial, \lambda) \) and \( S(\partial, \lambda) \) are uniquely determined by \( f(\partial, \lambda) \) and \( g(\partial, \lambda) \). Now we may assume that \( f(\partial, \lambda) \) and \( g(\partial, \lambda) \) are given by those polynomials in Theorem 2.4 with \( \deg g(\partial, \lambda) = \deg R(\partial, \lambda) \) due to (4.35), (4.36), and Lemma 4.3, and we may use (4.41) and (4.42) to define \( R(\partial, \lambda) \) and \( S(\partial, \lambda) \). After that we may then check if the quadruple so defined satisfies (4.37)\textendash (4.40). Again we have resorted to Maple V to perform this computation and below we will first write down these pairs of \( (f(\partial, \lambda), g(\partial, \lambda)) \) that are allowed and then discuss solutions to each such pair.

The admissible pairs are as follows.

1. \( m = 7 \).
   (i) \( \Delta - \Delta' = 6, \Delta' = (-5 \pm \sqrt{19})/2, f(\partial, \lambda) = f_1(\partial, \lambda), \) and \( g(\partial, \lambda) = 0 \).
   (ii) \( \Delta - \Delta' = 6, \Delta' + \frac{1}{2} = (-5 \pm \sqrt{19})/2, f(\partial, \lambda) = 0, g(\partial, \lambda) = f_1(\partial, \lambda) \) with \( \Delta' \) replaced by \( \Delta' + \frac{1}{2} \).

2. \( m = 6 \).
   (i) \( \Delta = 5, \Delta' = 0, f(\partial, \lambda) = f_6(\partial, \lambda), \) and \( g(\partial, \lambda) = 0 \).
   (ii) \( \Delta = 1, \Delta' = -4, f(\partial, \lambda) = f_6(\partial, \lambda), \) and \( g(\partial, \lambda) = 0 \).
   (iii) \( \Delta = \frac{1}{2}, \Delta' = -\frac{1}{2}, f(\partial, \lambda) = 0, g(\partial, \lambda) = f_6(\partial, \lambda) \).
   (iv) \( \Delta = \frac{1}{2}, \Delta' = -\frac{2}{2}, f(\partial, \lambda) = 0, g(\partial, \lambda) = f_6(\partial, \lambda) \).

3. \( m = 5, \Delta - \Delta' = 4, f(\partial, \lambda) = c_1 f_5(\partial, \lambda), \) and \( g(\partial, \lambda) = c_2 f_5(\partial, \lambda) \), where in \( g(\partial, \lambda) \Delta' \) is replaced by \( \Delta' + \frac{1}{2}(c_1, c_2) \neq (0, 0) \).

4. \( m = 4, \Delta - \Delta' = 3, f(\partial, \lambda) = c_1 f_4(\partial, \lambda), \) and \( g(\partial, \lambda) = c_2 f_4(\partial, \lambda), (c_1, c_2) \neq (0, 0) \).

5. \( m = 3, \Delta - \Delta' = 2, f(\partial, \lambda) = c_1 f_3(\partial, \lambda), \) and \( g(\partial, \lambda) = c_2 f_3(\partial, \lambda), (c_1, c_2) \neq (0, 0) \).

6. \( m = 2 \).
   (i) \( \Delta = 1, \Delta' = 0, f(\partial, \lambda) = c_1 \partial + \lambda^2, \) and \( g(\partial, \lambda) = 0, (c_1, c_2) \neq (0, 0) \).
   (ii) \( \Delta = \frac{1}{2}, \Delta' = -\frac{1}{2}, f(\partial, \lambda) = 0, g(\partial, \lambda) = c_1 \partial + \lambda^2, (c_1, c_2) \neq (0, 0) \).

7. \( m = 1 \).
In the case the unique solution is given by

\[ f(\tilde{\partial}, \lambda) = -(6\Delta' + 1)\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \Delta'\lambda^3), \]

\[ g(\tilde{\partial}, \lambda) = 4\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + (\Delta' + \frac{1}{2})\lambda^3), \]

\[ R(\tilde{\partial}, \lambda) = (1 - 2\Delta')\partial^2 - 8\Delta'\partial\lambda - 8\partial^3\lambda^2 + 16(5\Delta' + 3)\partial^2\lambda^3 + 8(15\Delta' + 4)\partial\lambda^4 + 8(7\Delta' + 2)\lambda^5. \]

(4) Solutions exist if and only if \( \Delta = 3 \) and \( \Delta' = 0 \) or \( \Delta = \frac{1}{2} \) and \( \Delta' = -\frac{1}{2}. \) In the first case we must have \( c_2 = 6c_1, \) while in the second case we must have \( c_1 = 6c_2. \) In the case \( \Delta = 3 \) and \( \Delta' = 0 \) the unique (up to a scalar) solution is

\[ f(\tilde{\partial}, \lambda) = \partial\lambda^2(\partial + \lambda), \quad R(\tilde{\partial}, \lambda) = -\partial^3 + 6\partial^2\lambda - 5\partial\lambda^2 - \lambda^3, \]

\[ g(\tilde{\partial}, \lambda) = 6\partial\lambda^2(\partial + \lambda), \quad S(\tilde{\partial}, \lambda) = \partial^4 + 2\partial^3\lambda + 4\partial^3\lambda. \]

In the case \( \Delta = \frac{1}{2} \) and \( \Delta' = -\frac{1}{2} \) the unique solution is

\[ f(\tilde{\partial}, \lambda) = 6\partial\lambda^2(\partial + \lambda), \quad R(\tilde{\partial}, \lambda) = -\partial^3 - 4\partial^2\lambda - 4\partial\lambda^3, \]

\[ g(\tilde{\partial}, \lambda) = \partial\lambda^2(\partial + \lambda), \quad S(\tilde{\partial}, \lambda) = \partial^3 + 2\partial^2\lambda - 6\partial\lambda^3 - 5\lambda^4. \]

(5) Solutions exist if and only if \((\Delta' + 1)(2\Delta' + 3)c_1 = \Delta'(2\Delta' + 1)c_2.\) The unique (up to a scalar) solution is

\[ f(\tilde{\partial}, \lambda) = \Delta'(2\Delta' + 1)\lambda^2(2\partial + \lambda), \]

\[ R(\tilde{\partial}, \lambda) = (\Delta' + 1)(2\Delta' + 3)\lambda^2(2\partial + \lambda), \]

\[ g(\tilde{\partial}, \lambda) = -3\partial^2 + 4\Delta'\partial\lambda + (2\Delta' + 3)\lambda^2, \]

\[ S(\tilde{\partial}, \lambda) = 3\partial^3 + 2\Delta'(3\partial^2\lambda + 2\Delta'(4\Delta' + 7)\partial\lambda^2 + 4\Delta'\lambda^3 + 4\Delta'\lambda^3). \]

(7i) Solutions exist if and only if \( c_1 = c_2. \) The unique (up to a scalar) solution is given by

\[ f(\tilde{\partial}, \lambda) = \lambda, \quad R(\tilde{\partial}, \lambda) = -\lambda, \quad c \in \mathbb{C}, \]

\[ g(\tilde{\partial}, \lambda) = \lambda, \quad S(\tilde{\partial}, \lambda) = \lambda + 2\Delta'\lambda + 2\lambda. \]

(8) Solutions exist if and only if \( c_1 = c_2. \) The unique (up to a scalar) solution is

\[ f(\tilde{\partial}, \lambda) = 1, \quad R(\tilde{\partial}, \lambda) = 0, \]

\[ g(\tilde{\partial}, \lambda) = 1, \quad S(\tilde{\partial}, \lambda) = 1. \]

We summarize the above discussion in the following theorem.
Theorem 4.2: Nontrivial extensions of the form (4.26) exist only if \( \alpha = \beta \) and \( \Delta = \Delta' = 0, 2, 3, 4 \). The following is a complete list of values of \( \Delta \) and \( \Delta' \) along with the quadruple of polynomials \( f(\delta, \lambda), g(\delta, \lambda), R(\delta, \lambda), \) and \( S(\delta, \lambda), \) whose nonzero scalar multiples give rise to nontrivial extensions (\( \delta = \delta + \alpha \)):

(i) \( \Delta = \Delta' \),

\[
\begin{align*}
f(\delta, \lambda) &= c_1 \lambda + c_2, \\
g(\delta, \lambda) &= c_1 \lambda + c_2, \\
R(\delta, \lambda) &= -c_3, \\
S(\delta, \lambda) &= c_3 (\delta + 2 \Delta' \lambda) + 2 c_1 \lambda + c_2,
\end{align*}
\]

with \( c_1, c_2, c_3 \in \mathbb{C} \) and \( (c_1, c_2, c_3) \neq (0, 0, 0) \).

(ii) \( \Delta - \Delta' = 2 \),

\[
\begin{align*}
f(\delta, \lambda) &= \Delta' (2 \Delta' + 1) \lambda^2 (2 \delta + \lambda), \\
R(\delta, \lambda) &= (\Delta' + 1) (2 \Delta' + 3) \lambda^2 (2 \delta + \lambda), \\
g(\delta, \lambda) &= -3 \delta^2 + 4 \Delta' \delta \lambda + (2 \Delta' + 3) \lambda^2, \\
S(\delta, \lambda) &= 3 \delta^3 + 2 (2 \Delta' + 3) \delta^2 \lambda + 2 \Delta' (4 \Delta' + 7) \delta \lambda^2 + 4 \Delta' (\Delta' + 2) \lambda^3.
\end{align*}
\]

(iii) \( \Delta = \frac{1}{2}, \Delta' = -\frac{1}{2} \),

\[
\begin{align*}
f(\delta, \lambda) &= 6 \delta \lambda^2 (\delta + \lambda), \\
R(\delta, \lambda) &= -\delta^3 - 4 \delta^2 \lambda + 4 \lambda^3, \\
g(\delta, \lambda) &= \delta \lambda^2 (\delta + \lambda), \\
S(\delta, \lambda) &= \delta^3 + 2 \delta^2 \lambda - 6 \delta \lambda^3 - 5 \lambda^4.
\end{align*}
\]

(iii') \( \Delta = 3, \Delta' = 0 \),

\[
\begin{align*}
f(\delta, \lambda) &= -6 \delta \lambda^2 (\delta + \lambda), \\
R(\delta, \lambda) &= -\delta^3 + \delta^2 \lambda + 5 \delta \lambda^2 - \lambda^3, \\
g(\delta, \lambda) &= 6 \delta \lambda^2 (\delta + \lambda), \\
S(\delta, \lambda) &= \delta^4 + 2 \delta^3 + 4 \delta \lambda^3.
\end{align*}
\]

(iv) \( \Delta - \Delta' = 4, \Delta' = (-7 \pm \sqrt{33})/4 \),

\[
\begin{align*}
f(\delta, \lambda) &= -(6 \Delta' + 1) \lambda^2 (4 \delta^3 + 6 \delta^2 \lambda - \delta \lambda^2 + \Delta' \lambda^3), \\
g(\delta, \lambda) &= 4 \lambda^2 (4 \delta^3 + 6 \delta^2 \lambda - \delta \lambda^2 + (\Delta' + \frac{1}{2}) \lambda^3), \\
R(\delta, \lambda) &= (2 \Delta' - 1) \delta^3 + 8 (2 \Delta' + 1) \delta^2 \lambda + 2 (2 \Delta' + 1) \delta \lambda^2 - 32 (2 \Delta' + 1) \delta \lambda^3 - (46 \Delta' + 13) \lambda^4, \\
S(\delta, \lambda) &= (1 - 2 \Delta') \delta^3 - 8 \Delta' \delta^2 \lambda - 8 \delta \lambda^3 + 16 (5 \Delta' + 3) \delta \lambda^2 + 8 (15 \Delta' + 4) \delta \lambda^3 + 8 (7 \Delta' + 2) \lambda^5.
\end{align*}
\]

The space of extensions is three-dimensional in the case (i), one-dimensional in the case (ii)–(iv), and trivial for all other values of \( \Delta \) and \( \Delta' \).

In conclusion we provide a formula to translate the extensions thus obtained into the language of conformal modules. As a vector space \( E^c \) is isomorphic to \( \mathbb{C} [t, t^{-1}, \xi] e^{-a t \omega(t^{1/2}) - \Delta} \oplus \mathbb{C} [t, t^{-1}, \xi] e^{-a t \omega(t^{1/2}) - \Delta} \) with \( p(\omega(t^{1/2}) - \Delta) = 1 \) and \( p(\omega(t^{1/2}) - \Delta) = 1 \), \( \mathbb{C} [t, t^{-1}, \xi] e^{-a t \omega(t^{1/2}) - \Delta} \) is an \( \mathfrak{g} \)-submodule of \( E^c \) and on the space \( \mathbb{C} [t, t^{-1}, \xi] e^{a t \omega(t^{1/2}) - \Delta} \) the action is given by \( (f(t) \in \mathbb{C} [t, t^{-1}], g(t) \in \mathbb{C} [t, t^{-1}] e^{-a t}) : 
\]
\[ D^{(t)}(t)\omega^{(1/2)-\Delta} = (f(t)g^\prime(t) + f^\prime(t)g(t))\omega^{(1/2)-\Delta} \]
\[ + \sum_{i,k} b_{ik}(-1)^i \left( \sum_{j=0}^{i} i^j \frac{f^{(k+i-j)}(t)}{j!} g^{(j)}(t) \right) \omega^{(1/2)-\Delta'}, \]
\[ D^{(t)}(t)\xi\omega^{(1/2)-\Delta} = (f(t)g^\prime(t) + (1-\Delta)f^\prime(t)g(t))\xi\omega^{(1/2)-\Delta} \]
\[ + \sum_{i,k} d_{ik}(-1)^i \left( \sum_{j=0}^{i} i^j \frac{f^{(k+i-j)}(t)}{j!} g^{(j)}(t) \right) \xi\omega^{(1/2)-\Delta'}, \]
\[ D^{(t)}(t)\omega^{(1/2)-\Delta} = (f(t)g^\prime(t) + (1-2\Delta)f^\prime(t)g(t))\xi\omega^{(1/2)-\Delta} \]
\[ + \sum_{i,k} d_{ik}(-1)^i \left( \sum_{j=0}^{i} i^j \frac{f^{(k+i-j)}(t)}{j!} g^{(j)}(t) \right) \xi\omega^{(1/2)-\Delta'}, \]
\[ D^{(t)}(t)\xi\omega^{(1/2)-\Delta} = -f(t)g(t)\omega^{(1/2)-\Delta} \]
\[ + \sum_{i,k} c_{ik}(-1)^i \left( \sum_{j=0}^{i} i^j \frac{f^{(k+i-j)}(t)}{j!} g^{(j)}(t) \right) \omega^{(1/2)-\Delta'}, \]

where \( f(\tilde{\sigma},\lambda) = \sum_{i,k} a_{ik}\tilde{\sigma}^k \), \( g(\tilde{\sigma},\lambda) = \sum_{i,k} b_{ik}\tilde{\sigma}^k \), \( R(\tilde{\sigma},\lambda) = \sum_{i,k} c_{ik}\tilde{\sigma}^k \lambda^k \), and \( S(\tilde{\sigma},\lambda) = \sum_{i,k} d_{ik}\tilde{\sigma}^k \lambda^k \). Here, as before \( h^{(i)}(t) \) denotes the \( i \)th derivative of the polynomial \( h(t) \) with respect to \( t \).

**Remark 4.1:** Of course extensions of the types
\[
0 \to N'(\alpha,\Delta') \to E \to N(\beta,\Delta) \to 0
\]
and
\[
0 \to N'(\alpha,\Delta') \to E \to N'(\beta,\Delta) \to 0
\]
are obtained from Theorems 4.1 and 4.2 by reversing parity.

**Remark 4.2:** We would like to make the following correction in Ref. 2. The second sentence after (5.5) should be replaced by: If \( \varphi^{1,0} = 0 \), then \( c_{ik} = 0 \) for \( k \geq 1 \). Thus \( L_{\lambda}u = (\partial + \beta + \Delta \lambda)u + f(\lambda)u' \), where \( u \to u' \) is a \( U \)-module isomorphism from \( U \) to \( V \). Since \( f(\lambda) \) must satisfy (3.2), we get from the list of Theorem 3.1 the following possibilities:

(i) \( f(\lambda) = a_0 + a_1 \lambda \) in the case \( \Delta = \tilde{\Delta} \), \( (a_0,a_1) \neq (0,0) \),
(ii) \( f(\lambda) = a \lambda^2 \) in the case \( \Delta = 1 \) and \( \Delta = 0, a \neq 0 \),
(iii) \( f(\lambda) = a \lambda^3 \) in the case \( \Delta = -\Delta = 2, a \neq 0 \).

These three examples should be added to the list in Proposition 5.3 and Theorem 5.1 in the case \( U \equiv V \).

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