A bistatic inverse scattering problem for electromagnetic waves

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In this paper we discuss a bistatic inverse scattering problem for Maxwell’s equations. We show that both the electric permittivity and magnetic permeability can be uniquely recovered from the knowledge of $S(s, \theta, \theta')$ for all $s \in \mathbb{R}$ and special pairs of orthonormal vectors $(\theta, \theta')$ provided that they are close to constants, where $S(s, \theta, \theta')$ is the scattering kernel associated with Maxwell’s equations. In other words, in this scattering experiment, measurements are made on a set of a priori arranged pairs of incoming and reflected directions. © 2000 American Institute of Physics.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

This paper is to investigate a special type of inverse scattering problem for Maxwell’s equations. Let $E(x,t)$ and $H(x,t)$ denote the electric and magnetic fields, respectively. The propagation of electromagnetic waves in an inhomogeneous medium is described by Maxwell’s equations, i.e.,

$$\epsilon \partial_t E = \nabla \times H, \quad \mu \partial_t H = -\nabla \times E$$  \hspace{1cm} (1.1)

and

$$\nabla \cdot (\epsilon E) = \nabla \cdot (\mu H) = 0,$$  \hspace{1cm} (1.2)

where $\epsilon = \epsilon(x) > 0$ is the electric permittivity and $\mu = \mu(x) > 0$ is the magnetic permeability. We assume, throughout this article, that

$$\epsilon(x) = \mu(x) = 1 \quad \text{for } |x| \geq \rho > 0.$$  \hspace{1cm} (1.3)

By comparing with the homogeneous Maxwell’s equations,

$$\partial_t E = \nabla \times H, \quad \partial_t H = -\nabla \times E$$

we can establish a scattering theory for Eqs. (1.1) and (1.2) by the Lax-Phillips theory.¹ Let $S(s, \theta, \theta')$ with $\theta, \theta' \in \mathbb{S}^2, s \in \mathbb{R}$ be the associated scattering kernel. It is a $3 \times 3$ matrix-valued distribution. Actually, the kernel $S(s-s', \theta, \theta')$ for $\theta \neq \theta'$ is the Schwartz kernel of the scattering operator conjugation with the Lax-Phillips modified Radon transform. The inverse scattering problem for Maxwell’s equations (1.1) and (1.2) consists of the determination of medium parameters $\epsilon(x)$ and $\mu(x)$ by $S(s, \theta, \theta')$ for a certain set of $(s, \theta, \theta')$. For example, if we take $\theta' = -\theta \in \mathbb{S}^2$ and $s \in \mathbb{R}$, then we have the inverse backscattering problem. In Ref. 2, the author considered this problem for the nonmagnetic medium, i.e., $\mu(x)$ is a constant. It was pointed out in Ref. 3 that knowing only the backscattering data is not enough to simultaneously determine

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both \( e(x) \) and \( \mu(x) \). A natural question arises: can one recover both parameters by knowing \( S(s, \theta, \theta') \) for \( s \in \mathbb{R} \) and a set of pairs of \( (\theta, \theta') \) with \( \theta' \neq -\theta \) (not arbitrary \( \theta \) and \( \theta' \))? We call this the **bistatic** inverse scattering problem. The term “bistatic” is commonly used in Radar community when transmitters and receivers are separate (see Ref. 4). On the other hand, the backscattering problem is called **monostatic** due to the fact that transmitters and receivers are located at the same positions. This article is devoted to solve this bistatic inverse problem for electromagnetic waves. It turns out one can show that both medium parameters \( e(x) \) and \( \mu(x) \) can be uniquely determined by \( S(s, \theta, \theta') \) for \( s \in \mathbb{R} \) and special pairs of \( (\theta, \theta') \) with \( \theta \perp \theta' \) provided that they are a priori close to constants. Before stating our main result, we want to clearly describe and depend smoothly on \( S \).

In the following, we will use the spherical coordinates with respect to \( h \) and \( v \). Therefore, intuitively, the amount of information we need to solve this bistatic problem is \( N \times 2 \) for all \( s \) and \( \theta \). In fact, for a given incoming direction \( \theta \), it suffices to measure at most two reflected directions \( \theta' \)'s which are perpendicular to \( \theta \). The main idea in solving the bistatic inverse scattering problem originates from the propagation property of electromagnetic waves in free space where the direction of propagation, \( E \) and \( H \) are mutually orthogonal. Different types of inverse problems for electromagnetic waves have been extensively studied recently. The interested reader is referred to the monograph5 and references therein.

In Sec. II, we will review the Lax–Phillips scattering theory for the inhomogeneous Maxwell’s equations (1.1) and (1.2) and derive a useful identity which plays a key role in the proof of main theorem. In Sec. III, we discuss the construction of scattering solutions for (1.1) and (1.2)
(defined in Sec. II) by using techniques from the geometrical optics. Finally, the proof of main theorem is thoroughly explained in Sec. IV. Throughout this paper, $C$ is a general constant. Its value may vary from line to line.

II. SCATTERING THEORY AND THE MAIN IDENTITY

The arguments in Ref. 2, Secs. II and III can be directly applied to the problem here without further modifications. So we will omit all proofs in this section. The reader is referred to Ref. 2 for details. From the compactness assumption of the inhomogeneity, the Lax–Phillips scattering theory can be easily applied to Maxwell’s equations (1.1) and (1.2) by combining results in Refs. 6 and 7. Note that the Lax–Phillips scattering theory for Maxwell’s equations in a nonmagnetic medium is also reviewed in Ref. 2. It is readily seen that the same arguments can be extended to deal with the case of inhomogeneous $\mu(x)$. Therefore, let $S$ be the scattering operator and $R$ be the Lax–Phillips modified Radon transform (or translation representation), then the Schwartz kernel of $RSR^{-1}$ is written as

$$\delta(s-s')\delta(\theta-\theta')I+S(s-s',\theta,\theta'),$$

where $I$ is the $3 \times 3$ identity matrix. Now the scattering kernel is defined by $S(s,\theta,\theta')$.

Next we will derive a formula for $S_1S_2$, where $S_i$ is the scattering kernel related to $(\epsilon_i,\mu_i), i=1,2$. Let $E(t,x,\theta)$ and $H(t,x,\theta)$ be two $3 \times 3$ matrix-valued distributions satisfying the following matrix equations:

$$\begin{align*}
\epsilon \partial_t E &= P(\nabla)H \\
\mu \partial_t H &= -P(\nabla)E
\end{align*}$$

(2.1)

with the behavior at $t=0$,

$$\begin{align*}
E &= \Theta \delta(t-x \cdot \theta) \\
H &= \Theta^2 \delta(t-x \cdot \theta)
\end{align*}$$

(2.2)

where $P(\nabla)$ is a matrix differential operator defined by

$$P(\nabla) = \begin{pmatrix}
0 & -\partial_3 & \partial_2 \\
\partial_3 & 0 & -\partial_1 \\
-\partial_2 & \partial_1 & 0
\end{pmatrix}, \quad \partial_j = \partial_{x_j}, j=1,2,3,$$

and

$$\Theta = \begin{pmatrix}
0 & -\theta_3 & \theta_2 \\
\theta_3 & 0 & -\theta_1 \\
-\theta_2 & \theta_1 & 0
\end{pmatrix}, \quad \theta=(\theta_1, \theta_2, \theta_3), |\theta|=1.$$

The existence of $E$ and $H$ is proved in Ref. 2. Now using exactly the same techniques from Ref. 2, one can show that (see Lemma 3.1 and Proposition 3.1 in Ref. 2 for the proof)

**Proposition 2.1**: Let $(E_i, H_i)$ be the solution of (2.1) and (2.2) associated with $(\epsilon_i, \mu_i), i=1,2$. Then,

$$(S_1-S_2)(s,\theta,\theta') = \frac{1}{8\pi^2} \int \int \{(\epsilon_1-\epsilon_2)E_1(t,x,-\theta')E_1(-s-t,x,\theta) \\
+ (\mu_1-\mu_2)H_1(t,x,-\theta')H_1(-s-t,x,\theta)\}dxdt,$$

(2.3)

where $A'$ denotes the transpose of the matrix $A$. The integral (2.3) is interpreted in the distributional sense.
Since \( q_\epsilon := \epsilon_1 - \epsilon_2 \) and \( q_\mu := \mu_1 - \mu_2 \) are compactly supported, if \( S_1(s, \theta, -\theta') = S_2(s, \theta, -\theta') \) for all \( s \in \mathbb{R} \) and for any \((\theta, \theta') \in \mathbb{S}^2 \times \mathbb{S}^2\), then we have from (2.3) that

\[
\int \int \{ q_\epsilon E_2(t, x, \theta, \vartheta) E_1(s-t, x, \vartheta) + q_\mu H_2(t, x, \vartheta) H_1(s-t, x, \vartheta) \} dx dt = 0, \tag{2.4}
\]

for all \( s \in \mathbb{R} \). Now let \((E(t, x, \theta, p), H(t, x, \theta, p))\) be the scattering solution of Maxwell’s equations (1.1) and (1.2), i.e., \((E, H)\) satisfies (1.1), (1.2) with

\[
\begin{align*}
\left\{ \begin{array}{l}
E |_{t=0} = \theta \times p \delta(t-x \cdot \vartheta), \\
H |_{t=0} = \theta \times (\theta \times p) \delta(t-x \cdot \vartheta),
\end{array} \right.
\end{align*}
\tag{2.5}
\]

where \( p \in \mathbb{R}^3 \) is a polarization vector. Here we observe that \( E(t, x, \theta, p) = E(t, x, \theta) p \) and \( H(t, x, \theta, p) = H(t, x, \theta) p \). Also, it should be noted that \( \Theta p = \theta \times p \) and \( \Theta^2 p = \theta \times (\theta \times p) \). Therefore, by multiplying \( p^\prime \) and \( p \) on the left and right of (2.4), respectively, we get that

\[
\int \int \{ q_\epsilon E_2(t, x, \theta, \vartheta) \cdot E_1(s-t, x, \vartheta) + q_\mu H_2(t, x, \theta, \vartheta) \cdot H_1(s-t, x, \vartheta) \} dx dt = 0, \tag{2.6}
\]

where \((E_1, H_1)\) is the scattering solution of Maxwell’s equations (1.1), (1.2) associated with \((\epsilon_i, \mu_i), i=1,2\) (for \( i=2, \vartheta \) is replaced by \( \vartheta' \)). The identity (2.6) will be used subsequently to prove the main theorem.

To motivate our approach, we consider the formal linearization of (2.6). In other words, we take \((E_1, H_1) = (\theta \times p, \theta \times (\theta \times p)) \delta(t-x \cdot \vartheta) \) and \((E_2, H_2) = (\theta \times p, \theta \times (\theta \times p)) \delta(t-x \cdot \vartheta') \) in (2.6) and obtain

\[
\int \{ q_\epsilon (\theta \times p) \cdot (\theta' \times p) + q_\mu (\theta \times (\theta \times p)) \cdot (\theta' \times (\theta \times p)) \} \delta(s-x \cdot (\vartheta + \vartheta')) dx = 0. \tag{2.7}
\]

Now we first let \( \omega \in \Gamma_\delta(\eta) \) and \( \theta(\omega), \theta'(\omega) \) be defined by (1.4), (1.5), respectively. Then we have that

\[
\theta \perp \theta', \tag{2.8}
\]

and

\[
\theta + \theta' = \sqrt{2} \omega \tag{2.9}
\]

for all \( \omega \in \Gamma_\delta(\eta) \). By taking \( p = \omega \) and using (2.8), we can easily obtain that

\[
(\theta \times p) \cdot (\theta' \times p) = -\frac{1}{2} \quad \text{and} \quad (\theta \times (\theta \times p)) \cdot (\theta' \times (\theta \times p)) = 0 \tag{2.10}
\]

for all \( \omega \in \Gamma_\delta(\eta) \). Hence, it follows from (2.7) that

\[
\int q_\epsilon \delta(s - \sqrt{2} x \cdot \omega) dx = 0 \tag{2.11}
\]

for all \( s \in \mathbb{R} \) and \( \omega \in \Gamma_\delta(\eta) \). Applying the Fourier transform on (2.11) with respect to \( s \), we get that

\[
\hat{q}_\epsilon(\xi) = 0, \quad \forall \frac{\xi}{|\xi|} \in \Gamma_\delta(\eta), \tag{2.12}
\]

where \( \hat{\cdot} \) is the Fourier transform in \( x \in \mathbb{R}^3 \). Now choosing \( \eta = e_1 \) and \( \eta = e_2 \) in (2.12), respectively, we conclude that
\[
\ddot{q}_\epsilon(\xi) = 0, \quad \forall \frac{\xi}{|\xi|} \in \Gamma_\delta(e_1) \cup \Gamma_\delta(e_2).
\]  

Let \( \delta > 0 \) is sufficiently small such that
\[
\mathbb{S} \setminus \mathcal{M} = \Gamma_\delta(e_1) \cup \Gamma_\delta(e_2),
\]
where \( \mathcal{M} = \{ \omega \in \mathbb{S}^2 : \text{ang}(\omega, e_i) = \pi/4 \text{ or } \text{ang}(\omega, e_i) = 3\pi/4, i = 1, 2 \} \) is a measure zero set. Here \( \text{ang}(u, v) \) denotes the angle between two unit vectors \( u \) and \( v \). Therefore, combining (2.13) and (2.14) yields
\[
\dot{q}_\epsilon(\xi) = 0 \quad \text{a.e.}
\]

Since \( q_\epsilon \) is compactly supported, (2.15) implies that \( q_\epsilon = 0 \), i.e. \( \epsilon_1 = \epsilon_2 \).

To get \( \mu_1 = \mu_2 \), i.e. \( q_\mu = 0 \), we take \( p(\omega) \) to be the unit normal vector to the plane spanned by \( \theta(\omega) \) and \( \theta'(\omega) \). With this choice of \( p \), we observe that
\[
(\theta \times p) \cdot (\theta' \times p) = 0 \quad \text{and} \quad (\theta \times (\theta \times p)) \cdot (\theta' \times (\theta' \times p)) = 1
\]
for all \( \omega \in \Gamma_\delta(\eta) \). Following the same procedures as above, we can conclude that \( \dot{q}_\mu(\xi) = 0 \) a.e. and hence \( q_\mu = 0 \).

### III. SCATTERING SOLUTIONS

In this section we would like to derive the scattering solution \((E, H)\) of Maxwell’s equations in a form of progressing waves expansion with remainders. That is, let \((E, H)\) satisfy
\[
\epsilon \partial_t E = \nabla \times H, \quad \mu \partial_t H = -\nabla \times E
\]
with
\[
E|_{t=0} = \theta \times p \delta(t-x \cdot \theta), \quad H|_{t=0} = \theta \times (\theta \times p) \delta(t-x \cdot \theta),
\]
then \((E, H)\) can be expressed as
\[
E(t, x, \theta, p) = \sum_{k=1}^1 E^{(k)}(x, \theta, p) h_k(t-\phi(x, \theta)) + E'(t, x, \theta, p)
\]
and
\[
H(t, x, \theta, p) = \sum_{k=1}^1 H^{(k)}(x, \theta, p) h_k(t-\phi(x, \theta)) + H'(t, x, \theta, p),
\]
where \( h_k(s) = s^k / k! \) for \( k \geq 0 \) and \( h_{-1}(s) = \delta(s) \). Notice that the divergence-free condition (1.2) is redundant for the scattering solution since this condition is satisfied at \( t=0 \). It is well-known that the phase function \( \phi(x, \theta) \) will satisfy the eikonal equation,
\[
|\nabla \phi(x, \theta)|^2 = \epsilon(x) \mu(x)
\]
with
\[
\phi(x, \theta)|_{x \cdot \theta < -p} = x \cdot \theta.
\]

In view of the smallness conditions (1.6), one can easily show that if \( \epsilon \) is taken sufficiently small, then \( \phi \) can be solved in \( B_{4p} \), a ball centered at the origin with radius \( 4p \), and satisfies
(see Ref. 8). Denote $T = p + T_0 \varepsilon$. Now we can establish the following result:

**Proposition 3.2:** Assume that $\varepsilon(x)$ and $\mu(x)$ satisfy (1.3), (1.6). Then there exists a constant $C > 0$ such that for all $|r| \leq 3T$ and $\theta \in \mathbb{S}^2$, the solution $(E, H)$ of (3.1) and (3.2) is of the form (3.3) and (3.4) and $E^{(k)}$, $H^{(k)}$, $k = -1, 0, 1$ and $E'$, $H'$ depend linearly (hence smoothly) on $p$ satisfying

$$\|E^{(-1)} - (\theta \times p)\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(-1)} - \theta \times (\theta \times p)\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.6)$$

$$\|E^{(0)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(0)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.7)$$

$$\|E^{(1)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(1)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.8)$$

$$\|E''\|_{L^\infty} + \|\partial_r E''\|_{L^2} \leq C \varepsilon, \|H''\|_{L^\infty} + \|\partial_r H''\|_{L^2} \leq C \varepsilon, \quad (3.9)$$

and

$$E'(t, x, \theta, p) = H'(t, x, \theta, p) = 0 \quad \text{for} \quad t \leq -T. \quad (3.10)$$

**Proof:** Let $(E, H)$ be the solution of (2.1) and (2.2). Then from Ref. 2 (see Ref. 2, Corollary 4.1) we obtain that if $\varepsilon$ is sufficiently small, then there exists a constant $C > 0$ such that for $|r| \leq 3T$ and all $\theta \in \mathbb{S}^2$, $(E, H)$ satisfies

$$E(t, x, \theta) = \sum_{k=-1}^{1} E^{(k)}(x, \theta) h_k(t - \phi(x, \theta)) + E'(t, x, \theta)$$

and

$$H(t, x, \theta) = \sum_{k=-1}^{1} H^{(k)}(x, \theta) h_k(t - \phi(x, \theta)) + H'(t, x, \theta)$$

with the following estimates:

$$\|E^{(-1)} - O\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(-1)} - O\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.6)$$

$$\|E^{(0)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(0)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.7)$$

$$\|E^{(1)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \|H^{(1)}\|_{W^{3,\infty}(B_{4r} \times S^2_\theta)} \leq C \varepsilon, \quad (3.8)$$

and

$$\|E''\|_{L^\infty} + \|\partial_r E''\|_{L^2} \leq C \varepsilon, \|H''\|_{L^\infty} + \|\partial_r H''\|_{L^2} \leq C \varepsilon. \quad (3.9)$$

As explained before, Corollary 4.1 in Ref. 2 can be extended easily to cover the case where the magnetic permeability is not a constant. Now, by noting $E = E \rho$ and $H = H \rho$, we get that $E^{(k)} = E^{(k)} \rho$, $H^{(k)} = H^{(k)} \rho$ for $k = -1, 0, 1$ and $E' = E' \rho$, $H' = H' \rho$. Thus, we immediately have the required estimates (3.6)–(3.9). The property (3.10) is obvious. \(\square\)

**IV. PROOF OF MAIN THEOREM**

In this section we will use the identity (2.6) to prove the main theorem. To begin with, let us choose an appropriate cut-off function. Let $\chi_\eta(\omega) \in C^\infty(\mathbb{S}^2)$, where $\eta \in \mathbb{S}^2$ is a fixed vector, so that
\[ \chi_\eta(\omega) = \begin{cases} 0 & \text{if } \omega \in O_{g_2}(\eta) \\ 1 & \text{if } \omega \in S^3 \setminus O_{g_2}(\eta) \end{cases} \]

where \( O_{g_1}(\eta) = \{ \omega \in S^3 : \text{ang}(\omega) < \delta \text{ or } \text{ang}(\omega - \eta) < \delta \} \). Let \( \omega \in \text{supp } \chi_\eta \) be given and \( \theta(\omega), \theta'(\omega) \in S^2 \) be defined in (1.4), (1.5), respectively. As noted before, \( \theta(\omega) \) and \( \theta'(\omega) \) are smooth functions of \( \omega \in \text{supp } \chi_\eta \). Actually, we need to exclude a measure zero set from \( \text{supp } \chi_\eta \) when we consider \( \omega \in \text{supp } \chi_\eta \). However, since we are mainly dealing with integrals in this section, we can ignore the effect of this measure zero set. Thus, for simplicity, we will state all following results without explicitly indicating "a.e." Now let \( p \in S^2 \) be a polarization vector and denote \( \theta \times p = \theta_p , \theta' \times p = \theta'_p , \theta \times (\theta \times p) = \tilde{\theta}_p , \) and \( \theta' \times (\theta' \times p) = \tilde{\theta}'_p \). In the following, we will take different \( p's \) and \( \eta's \) as we did in Sec. II.

First of all, let \( p = \omega \), then the scattering solution \((E, H)\) constructed in Proposition 3.2 satisfies

\[
\|[E^{(1)} - \theta_p(\omega)]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon \|[H^{(1)} - \tilde{\theta}_p(\omega)]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon ,
\]

\[
\|[E^{(0)}]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon \|[H^{(0)}]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon ,
\]

\[
\|[E^{(1)}]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon \|[H^{(1)}]\|_{W^{1/2}(B_{4\rho} \times S^2)} \leq C_\varepsilon ,
\]

and

\[
\|[E']\|_{L^\infty} + \|\partial_r E'\|_{L^2} \leq C_\varepsilon \|[H']\|_{L^\infty} + \|\partial_r H'\|_{L^2} \leq C_\varepsilon ,
\]

where all estimates above are valid for \( \omega \in \text{supp } \chi_\eta \). Note that the constant \( C \) in (4.1)\textendash(4.4) may depend on \( \delta \), but it is a fixed constant as long as \( \delta \) is fixed. Denote \((E_1, H_1)\) and \((E_2, H_2)\) the scattering solutions described above associate with \((\epsilon_1, \mu_1)\) and \((\epsilon_2, \mu_2)\), respectively. Let us set \( \phi(x, \omega) = \phi_1(x, \theta(\omega)) + \phi_2(x, \theta'(\omega)) \), \((E_1, H_1)(t, x, \theta(\omega), p(\omega)) = (E_1, H_1)(t, x, \omega) \), and \((E_2, H_2)(t, x, \theta'(\omega), p(\omega)) = (E_2, H_2)(t, x, \omega) \). Now in view of the Parseval's formula for the Radon transform \( R, \|\partial_r R f\|_{L^2(R^3)} = 4 \pi \|f\|_{L^2(R^3)} \), we substitute \((E_i, H_i), i = 1, 2\) into the identity (2.6), differentiate (2.6) in \( s \) and multiply the new identity by \( \chi_\eta(\omega) \). Then we get that

\[
-\partial_s \chi_\eta \int q_s E_1^{(-1)} E_2^{(-1)} \delta(s - \phi) \, dx = T_1 + T_2 + T_3 ,
\]

where

\[
T_1 = \partial_s \chi_\eta \int q_\mu H_1^{(-1)} H_2^{(-1)} \delta(s - \phi) \, dx ,
\]

\[
T_2 = \chi_\eta \int \{ q_s (E_1^{(-1)} E_2^{(0)} + E_1^{(0)} E_2^{(-1)}) + q_\mu (H_1^{(-1)} H_2^{(0)} + H_1^{(0)} H_2^{(-1)}) \} \delta(s - \phi) \, dx ,
\]

\[
T_3 = \chi_\eta \int \{ q_s (E_1^{(-1)} E_2^{(1)} + E_1^{(1)} E_2^{(-1)}) + q_\mu (H_1^{(-1)} H_2^{(1)} + H_1^{(1)} H_2^{(-1)}) + H_1^{(1)} H_2^{(-1)}) \} h_0(s - \phi) \, dx + \chi_\eta \int \{ q_s (E_1^{(0)} E_2^{(1)} + E_1^{(1)} E_2^{(0)}) + q_\mu (H_1^{(0)} H_2^{(1)} + H_1^{(1)} H_2^{(0)}) \}
\]

\[
+ H_1^{(1)} H_2^{(-1)}) \} h_0(s - \phi) \, dx + \chi_\eta \int \{ q_s (E_1^{(1)} E_2^{(-1)} + q_\mu (H_1^{(1)} H_2^{(-1)})) h_1(t - \phi_1) + H_1^{(1)} H_2^{(-1)}) \} h_1(s - \phi_1) \, dx + \chi_\eta \int \{ q_s (E_1^{(1)} E_2^{(-1)} + q_\mu (H_1^{(1)} H_2^{(-1)})) h_1(t - \phi_1)
\]

\[
\times h_0(s - t - \phi_2) \, dt \, dx + \chi_\eta \int \{ q_s (E_1^{(-1)} \partial_s E_2^{(-1)} + q_\mu (H_1^{(-1)} H_2^{(-1)})) h_1(s - \phi_2) + q_\mu (H_1^{(-1)} H_2^{(-1)})) \} \}
\]
satisfy, we adopt a notation from Ref. 9. We say that
\begin{align}
\partial_t H_2^2(s-\phi_1) + H_2^{-1}(s-\phi_2)) dx + \chi \int \left\{ q_\epsilon E_2^{(0)} \cdot E_2^r(s-\phi_1) + E_2^{(0)} \cdot E_2^r(s-\phi_2) \right\} \nonumber \\
+ q_\mu (H_2^{(0)} \cdot H_2^r(s-\phi_1) + H_2^{(0)} \cdot H_2^r(s-\phi_2)) dx + \chi \int \left\{ q_\epsilon E_1^{(1)} \cdot \partial_t E_1^r(s-t) \right\} \nonumber \\
\cdot \partial_t H_2^1(s-t)) h_1(t-\phi_1) dt dx + \chi \int \left\{ q_\epsilon E_1^{(1)} \cdot \partial_t E_1^r(s-t) + q_\mu H_2^{(1)} \cdot \partial_t H_2^1(s-t) \right\} \nonumber \\
\times h_1(t-\phi_2) dt dx + \chi \int \left\{ q_\epsilon E_1^{(1)} \cdot \partial_t E_2^r(s-t) + q_\mu H_2^{(1)} \cdot \partial_t H_2^1(s-t) \right\} dt dx . \tag{4.6}
\end{align}

Since \( q_\epsilon \) vanishes outside of \( B_\rho \), the left-hand side of (4.5) is supported on \( |s| < 2T \), where \( T \) is given in Proposition 3.2. Hence, it is also true for the right-hand side of (4.5). In view of this fact, we will square integrate both sides of (4.5) over the region \([ -2T,2T ] \times S^2 \). We first look at the left-hand side of (4.5). Observe that
\begin{align}
\| \partial_x \chi_{g}^{(1)}(\omega) \int q_\epsilon(x) a(x,\omega) \delta(s-\phi) dx \|_{L^2([-2T,2T] \times S^2)} & = \| \partial_x \chi_{g}^{(1)}(\omega) \int q_\epsilon(x) a(x,\omega) \delta(s-\phi) dx \|_{L^2(\mathbb{R} \times S^2)} \tag{4.7} \\
& = \frac{1}{\sqrt{2\pi}} \| \chi_{g}^{(1)}(\omega) \int e^{i\lambda \phi(x,\omega)} q_\epsilon(x) a(x,\omega) dx \|_{L^2(\mathbb{R} \times S^2)} ,
\end{align}
where \( a(x,\omega) = E_2^{(-1)}(x,\omega) \cdot E_2^{(-1)}(x,\omega) \). Since \( q_\epsilon(x) \) is real-valued, by setting \( \xi = \lambda \omega, \lambda > 0, \omega \in S^2 \), the integral in (4.7) can be rewritten as
\begin{align}
\| \partial_x \chi_{g}^{(1)}(\omega) \int q_\epsilon(x) a(x,\omega) \delta(s-\phi) dx \|_{L^2([-2T,2T] \times S^2)} & = \frac{1}{\sqrt{2\pi}} \| P_\epsilon q_\epsilon \|_{L^2(\mathbb{R} \times S^2)} ,
\end{align}
where
\begin{align}
P_\epsilon q_\epsilon(\xi) = \int e^{i\phi(x,\xi)} \chi_{g}^{(1)}(\xi) a(x,\xi) q_\epsilon(x) dx . \tag{4.8}
\end{align}
Here \( \chi_{g}^{(1)}(\xi), \phi(x,\xi), \) and \( a(x,\xi) \) have been extended to \( \xi \in S^2 \) by defining \( \chi_{g}^{(1)}(\xi) = \chi_{g}^{(1)}(\xi/|\xi|) \), \( \phi(x,\xi) = |\xi| \phi(x,|\xi|) \), and \( a(x,\xi) = a(x,|\xi|) \) for \( \xi \neq 0 \). To state what estimates \( \phi \) and \( a \) will satisfy, we adopt a notation from Ref. 9. We say that \( a = a(x,\xi) \in S_k^{m} \) iff there exists a constant \( C > 0 \) such that
\begin{align}
|\partial_x^\alpha \partial_{\xi}^\beta a(x,\xi)| \leq C |\xi|^{m-|\beta|} , \quad |\alpha| + |\beta| \leq k , \quad \text{for} \quad x \in B_\rho , \xi \in \mathbb{R}^3 \setminus \{0\} . \tag{4.9}
\end{align}
The optimal constant in (4.9) defines a norm in \( S_k^{m} \) and \( a = O(\epsilon) \) in \( S_k^{m} \) means that \( a \) satisfies (4.9) with \( C = O(\epsilon) \). Now it follows from (3.5) that
\begin{align}
\phi(x,\xi) = \sqrt{2} x \cdot \xi + O(\epsilon) \quad \text{in} \quad S_{11}^{1} \quad \text{for} \quad \xi \in \text{supp} \ \chi_{g}^{(1)}(\xi) , \tag{4.10}
\end{align}
and from the first part of (4.1) that
\begin{align}
a(x,\xi) = \theta_\rho(\xi/|\xi|) \cdot \theta_\rho'(\xi/|\xi|) + O(\epsilon) = -1/2 + O(\epsilon) \quad \text{in} \quad S_{10}^{0} \quad \text{for} \quad \xi \in \text{supp} \ \chi_{g}^{(1)}(\xi) . \tag{4.11}
\end{align}
Now we want to estimate \( P_\epsilon q_\epsilon \). We can show that
Proposition 4.3: Let $\varepsilon$ be sufficiently small, then there exists an $\varepsilon$-independent constant $C > 0$ such that
\[
\|P_{a}q\|_{L^{2}(\mathbb{R}^{3})} \geq \left( \frac{1}{128} \right)^{1/4} \left( \frac{1}{2\pi} \right)^{3/2} \|q\|_{L^{2}(\Omega_{\delta}(\eta))} - C\varepsilon\|q\|_{L^{2}(\mathbb{R}^{3})},
\]
where $\Omega_{\delta}(\eta) = \{ \xi \in \mathbb{R}^{3} : |\xi| \in \Gamma_{\delta}(\eta) \}$.

Proof: Consider
\[
P_{a}P_{a}q_{a}(y) = \int_{0}^{1} e^{i(\phi(y, \xi) - \phi(x, \xi))} \chi_{a}(\xi) a(y, \xi) a(x, \xi) q_{a}(x) dxd\xi.
\]

Now we set
\[
\phi(y, \xi) - \phi(x, \xi) = \sqrt{2}(y - x) \cdot \zeta(y, x, \xi),
\]
where
\[
\zeta(y, x, \xi) = \frac{1}{\sqrt{2}} \int_{0}^{1} (\nabla_{x} \phi)(x + t(y - x), \xi) dt.
\]
It should be noted that for $\xi \in \text{supp} \chi_{a}(\xi), \zeta$ is homogeneous of degree one in $\xi$ and $\zeta = \xi + O(\varepsilon)$ in $S_{10}^{1}$. Thus, the equation $\zeta = \xi(x, y, \xi)$ can be solved for $\xi$, when $\varepsilon$ is sufficiently small and $\xi \in \text{supp} \chi_{a}(\xi), (x, y) \in B_{\delta}^{\text{supp}}$. Moreover, the solution $\xi = \xi(x, y, \xi)$ satisfies $\xi = \xi + O(\varepsilon)$ in $S_{10}^{1}$. By performing a change of coordinates $\xi \rightarrow \zeta$, we have that
\[
P_{a}P_{a}q_{a}(y) = \int_{0}^{1} e^{i(\zeta(x, y, \zeta))} \chi_{a}(\xi) a(x, \zeta) q_{a}(x) dxd\zeta,
\]
where
\[
\chi_{a}(x, y, \zeta) = \chi_{a}(\xi(x, y, \zeta)) a(y, \zeta) a(x, \zeta) J(x, y, \zeta)
\]
and $J(x, y, \zeta)$ is the Jacobian of the transform $\xi \rightarrow \zeta$. Now it is readily to check that
\[
J = 1 + O(\varepsilon) \text{ in } S_{9}^{0}
\]
and from (4.11) we get that
\[
\chi_{a}(x, y, \zeta) = \frac{1}{4} \chi_{a}(\xi) + O(\varepsilon) \text{ in } S_{9}^{0}
\]
(4.12)

To continue the proof, we recall a result in Ref. 8.

Lemma 4.1: (Ref. 8) Let the operator $A$ be defined by
\[
Af(y) = \int_{0}^{1} e^{i(x - y \cdot \zeta)} q(x, y, \zeta) f(y) dxd\zeta.
\]
Assume that
\[
\sum_{|\alpha| + |\beta| \leq m} \int_{0}^{1} |\partial_{x}^{\alpha} \partial_{y}^{\beta} q(x, y, \zeta)| dxdy \leq M
\]
(4.13)
for $m \geq 7$. Then $A : L^{2}(\mathbb{R}^{3}) \rightarrow L^{2}(\mathbb{R}^{3})$ is a bounded operator with the norm $\leq CM$ for some constant $C > 0$, i.e.,

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\[ \| Af \|_{L^2(\mathbb{R}^3)} \lesssim CM \| f \|_{L^2(\mathbb{R}^3)} \]

for all \( f \in L^2(\mathbb{R}^3) \).

Now let \( \sigma(x) \in C_c^\infty(\mathbb{R}^3) \) be a cutoff function with \( \sigma = 1 \) on \( \overline{B_\rho} \) and \( \sigma = 0 \) for \( x \in \mathbb{R}^3 \setminus B_{2\rho} \), then (4.12) implies that \( \sigma(y)(\overline{a(x,y,\xi)} - \frac{1}{4} \chi_y^2(\xi)) \sigma(x) \) satisfies (4.13) with \( M = O(\varepsilon) \). Therefore, let \( \overline{\chi_y}(\xi) = \chi_y(\xi/\sqrt{2}) \), then we get from Lemma 4.1 that

\[
\begin{align*}
\| P_b q_d \|_{L^2(\mathbb{R}^3)}^2 &- \frac{1}{8\sqrt{2}} \left( \frac{1}{2\pi} \right)^3 \| \overline{\chi_y} \overline{a} \|_{L^2(\mathbb{R}^3)}^2 \\
&= \left| \left( \int \int e^{i\sqrt{2}(\xi - \eta) \cdot \sigma(y)(\overline{a(x,y,\xi)} - \frac{1}{4} \chi_y^2(\xi)) \sigma(x)q_d(x)dx \right) \right| (4.14) \\
&\leq C\varepsilon \| q_d \|_{L^2(\mathbb{R}^3)}^2.
\end{align*}
\]

Now in view of the structure of \( \chi_y \), this proposition is an easy consequence of (4.14).

From Proposition 4.3, it is obvious that

\[
C\| \overline{\chi_y} \|_{L^2(\Omega, \eta)} - C\varepsilon \| q_d \|_{L^2(\mathbb{R}^3)} \leq \| \sigma \chi_y(\omega) \int q_d(x)a(x,\omega)\delta(s - \phi)dx \|_{L^2([-2T,2T] \times S^2_\varepsilon)}.
\]

(4.15)

We now turn our attention to the right-hand side of (4.5). Let us first take care of the term \( T_1 \). As before, let \( b(x,\omega) = H^{(-1)}_1(x,\omega) : H^{(-1)}_2(x,\omega) \) be a cutoff function, then we can get that

\[
\| \sigma \chi_y(\omega) \int q_d(x)b(x,\omega)\delta(s - \phi)dx \|_{L^2([-2T,2T] \times S^2_\varepsilon)} = \frac{1}{\sqrt{\pi}} \| P_b q_d \|_{L^2(\mathbb{R}^3)}.
\]

where the operator \( P_b \) is defined as in (4.8) with \( a \) being replaced by \( b \). Here \( b(x,\xi) \) is also extended to \( \xi \notin S^2 \) with \( b(x,\xi) = b(x,\xi|\xi|) \). Now from the second part of (4.1), the amplitude \( b(x,\xi) \) satisfies

\[
b(x,\xi) = \overline{\tilde{\delta}_p}(|\xi|) \cdot \overline{\tilde{\eta}_p}(|\xi|) + O(\varepsilon) = O(\varepsilon) \text{ in } S^0_\varepsilon \text{ for } \xi \in \text{ supp } \chi_y(\xi). \tag{4.16}
\]

By mimicking the proof of Proposition 4.3, we can easily show that

\[
\| P_b q_d \|_{L^2(\mathbb{R}^3)} \leq C\varepsilon \| q_d \|_{L^2(\mathbb{R}^3)}.
\]

Indeed, combining Lemma 4.1 and the estimate (4.16) and using the same change of coordinates (i.e., \( \xi \rightarrow \xi \)), we can get that

\[
\| P_b q_d \|_{L^2(\mathbb{R}^3)} \} \leq C\varepsilon \| q_d \|_{L^2(\mathbb{R}^3)}.
\]

which obviously implies (4.17). Now the estimate (4.17) leads to

\[
\| T_1 \|_{L^2([-2T,2T] \times S^2_\varepsilon)} \leq C\varepsilon \| q_d \|_{L^2(\mathbb{R}^3)}.
\]

(4.18)

To deal with the term \( T_2 \), we use estimates (4.1) and (4.2) and the fact that \( q_e, q_d \) are compactly supported to derive that
\[ \|T_2\|_{L^2([-2T,2T] \times S^2)} \leq C \varepsilon \left( \left\| \chi_\eta \int |q_\mu \delta(s - \phi) dx \right\|_{L^2(\mathbb{R} \times S^2)} + \left\| \chi_\eta \int |p_\mu \delta(s - \phi) dx \right\|_{L^2(\mathbb{R} \times S^2)} \right) \]

where \( P_1 \) is defined in (4.8) with \( a = 1 \). Using the same techniques as above, we can show that

\[ \|P_1 g\|_{L^2(\mathbb{R}^3)} \leq C \varepsilon \|g\|_{L^2(\mathbb{R}^3)} \text{ for } g \in L^2(\mathbb{R}^3), \text{ supp } g \subset B_\rho. \]

Therefore, we have that

\[ \|T_2\|_{L^2([-2T,2T] \times S^2)} \leq C \varepsilon (\|q_\mu\|_{L^2(\mathbb{R}^3)} + \|p_\mu\|_{L^2(\mathbb{R}^3)}). \tag{4.19} \]

Finally, we will handle the term \( T_3 \). To this end, we observe that \( T_3 \) can be rewritten as

\[ T_3 = \int K_E(s, \omega, x) q_\varepsilon(x) dx + \int K_H(s, \omega, x) q_\mu(x) dx, \]

where for \( |s| \leq 2T \) and \( x \in B_\rho \),

\[ K_E(s, \omega, x) = \chi_\eta(E_1^{(-1)} \cdot E_2^{(1)} + E_1^{(0)} \cdot E_2^{(0)} + E_1^{(-1)} \cdot E_2^{(1)} + E_1^{(1)} \cdot E_2^{(1)}) \]

\[ \times h_1(s - \phi) + \chi_\eta(\int_{-T}^{3T} (E_1^{(1)} \cdot E_2^{(1)}) h_1(t - \phi_1) h_0(s - t - \phi_2) dt + \chi_\eta(\int_{-T}^{3T} (E_1^{(-1)} \cdot \delta_\mu E_2^{(1)}(s - \phi_1) + \delta_\mu E_1^{(-1)} \cdot \delta_\mu E_2(s - \phi_2) + E_1^{(0)} \cdot E_2^{(0)}(s - \phi_1)

\[ + E_1^{(1)} \cdot E_2^{(-1)}(s - \phi_2) + \chi_\eta(\int_{-T}^{\rho + 2T} (E_1^{(1)} \cdot \delta_\mu E_2^{(1)}(s - t) h_1(t - \phi_1) dt + \chi_\eta(\int_{-T}^{\rho + 2T} E_1^{(1)} \cdot \delta_\mu E_2^{(1)}(s - t) dt \]

and \( K_H(s, \omega, x) \) is given as (4.20) with \( E \) being replaced by \( H \). Since \( |s| \leq 2T \) and \( x \in B_\rho \), we have that \( |s - \phi_1| \leq 3T \) and \( |s - \phi_2| \leq 3T \) for \( \omega \in \text{supp } \chi_\eta(\omega) \). Moreover, if \( -T \leq -\rho \leq \rho - 2T \), then we have \( -5T \leq s - t \leq 3T \). Hence we see that the argument of \( E'_i(t) \) and \( H'_i(t) \), \( i = 1, 2 \) in the above integral lies inside the estimate region of Proposition 3.2. Now using all estimates in Proposition 3.2 we can derive that

\[ \int_{-2T}^{2T} \int_{S^2} \int_{B_\rho} |K_E(s, \omega, x)|^2 + |K_H(s, \omega, x)|^2 dx d\omega ds \leq (C \varepsilon)^2. \tag{4.21} \]

It follows from (4.21) that

\[ \|T_3\|_{L^2([-2T,2T] \times S^2)} \leq C \varepsilon (\|q_\mu\|_{L^2(\mathbb{R}^3)} + \|p_\mu\|_{L^2(\mathbb{R}^3)}). \tag{4.22} \]

Combining the estimates (4.15), (4.18), (4.19), (4.22), we immediately have that

\[ C \|q_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C \varepsilon (\|q_\mu\|_{L^2(\mathbb{R}^3)} + \|p_\mu\|_{L^2(\mathbb{R}^3)}). \tag{4.23} \]

Now let the fixed constant \( \delta > 0 \) be small such that

\[ S^2 = \Gamma_\varepsilon(e_1) \cup \Gamma_\varepsilon(e_2) \quad (a.e.) \]
and therefore
\[ \mathbb{R}^3 = \Omega_{g}(e_1) \cup \Omega_{g}(e_2) \quad \text{(a.e.)}. \] (24)

Thus, in view of (4.24), we obtain that
\[ C \| q_e \|_{L^2(\mathbb{R}^3)} = C \| \hat{q}_e \|_{L^2(\mathbb{R}^3)} \leq C(\| \hat{q}_e \|_{L^2(\Omega_{g}(e_1))} + \| \hat{q}_e \|_{L^2(\Omega_{g}(e_2))}) \leq C e(\| q_e \|_{L^2(\mathbb{R}^3)} + \| q_{\mu} \|_{L^2(\mathbb{R}^3)}). \] (4.25)

Next we want to prove the following estimate similar to (4.25),
\[ C \| q_{\mu} \|_{L^2(\mathbb{R}^3)} \leq C e(\| q_e \|_{L^2(\mathbb{R}^3)} + \| q_{\mu} \|_{L^2(\mathbb{R}^3)}). \] (4.26)

To this end, we choose \( p(\omega) \in S^2 \) with \( p(\omega) \perp \theta(\omega) \) and \( p(\omega) \perp \theta'(\omega) \). Thus, we have that
\[ (\theta \times p) \cdot (\theta' \times p) = 0 \quad \text{and} \quad (\theta \times (\theta \times p)) \cdot (\theta' \times (\theta' \times p)) = 1 \] (4.27)
for all \( \omega \in \Gamma_p(\eta) \). We now rearrange the formula (4.5) such that
\[ -\partial_s \chi_\eta \int q_{\mu} \overline{H_1^{(-1)}} : \overline{H_2^{(-1)}} \delta(s - \phi) \, dx = T_1 + T_2 + T_3, \]
where
\[ T_1 = -\partial_s \chi_\eta \int q_{\mu} \overline{E_1^{(-1)}} : \overline{E_2^{(-1)}} \delta(s - \phi) \, dx \]
and \( T_2 \) and \( T_3 \) are defined as in (4.6). With this choice of \( p \), the scattering solutions \( (E_1, H_1) \) and \( (E_2, H_2) \) constructed in Proposition 3.2 with respect to \( (e_1, \mu_1) \) and \( (e_2, \mu_2) \) will satisfy the same estimates (4.1)–(4.4). From these estimates and (4.27), we observe that
\[ H_1^{(-1)} : H_2^{(-1)} = 1 + O(\varepsilon) \quad \text{in} \quad S^0 \] (4.28)
and
\[ E_1^{(-1)} : E_2^{(-1)} = O(\varepsilon) \quad \text{in} \quad S^0 \] (4.29)
for \( \xi \in \text{supp} \chi_\eta(\xi) \). Therefore, by noting (4.28) and (4.29), we can go over the above arguments again and obtain (4.26) (also taking \( \eta = e_1 \) and \( e_2 \), respectively). Combining (4.25) and (4.26) yields
\[ C(\| q_e \|_{L^2(\mathbb{R}^3)} + \| q_{\mu} \|_{L^2(\mathbb{R}^3)}) \leq C e(\| q_e \|_{L^2(\mathbb{R}^3)} + \| q_{\mu} \|_{L^2(\mathbb{R}^3)}). \]

Thus, by taking \( e \) to be sufficiently small, we immediately have that \( q_e = 0 \) and \( q_{\mu} = 0 \), i.e., \( \varepsilon_1 = \varepsilon_2 \) and \( \mu_1 = \mu_2 \).

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