Fractional exclusion statistics of two-species gases

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The thermodynamics of two-species gases described by the Haldane exclusion statistics is investigated. I use the high-temperature perturbation to find the third Virial coefficient for the gas in an arbitrary dimension. The perturbative result indicates that the two-dimensional thermodynamic potential becomes the linear function of the element of the Haldane statistics matrix, if the matrix is symmetric and particle mass is species independent. I also present a nonperturbative method to prove this factorizable property and an exact expression of the thermodynamic potential is suggested. [S0163-1829(99)10847-6]

I. INTRODUCTION

The system of the particles obeying the fractional exclusion statistics of anyons appears naturally in the fractional quantized Hall effect and spin-1/2 antiferromagnetic chain. It had also been proposed that they may play important roles in the high-temperature superconductor. The thermodynamic behavior of the system had been studied by several authors and the Virial coefficient of the free anyon was calculated therein.

Wu and others had derived the distribution function of occupation number for the gases obeying the fractional exclusion statistics. They formulated the theory of quantum statistical mechanics and studied the thermodynamics of the ideal gas. In a Letter I have evaluated the high-temperature expansion of the energy $E(\alpha, N)$ to the seventh order. The result indicates that, in two dimensions, the N-particle thermodynamic quantities $Q_{\alpha}(N)$ can be factorized in terms of these in the ideal boson ($\alpha = 0$) and in the ideal fermion gases ($\alpha = 1$), i.e.,

$$Q_{\alpha}(N) = (1 - \alpha)Q_0(N) + \alpha Q_1(N),$$

where

$$Q_0(N) = -NkT \sum_{l=0}^{d} \frac{B_l}{(l+1)!} \left( \frac{N\lambda^2}{V} \right)^l,$$

$$Q_1(N) = -NkT \sum_{l=0}^{d} (-1)^l \frac{B_l}{(l+1)!} \left( \frac{N\lambda^2}{V} \right)^l.$$

Note that $\lambda$ is the thermal wavelength and $B_l$ are the Bernoulli number with $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_{2l+1} = 0$, for $l \geq 1$. There I also presented a nonperturbative proof to show this factorizable property.

According to the original definition, the fractional exclusion statistics may exist in any spatial dimension, in contrast to the anyons which connect to the braid group and can only make sense in two spatial dimensions. The state-counting definition of the fractional exclusion statistics also allows the exclusion to occur between different species. In papers, the quasielectrons and the quasiholes in the fractional quantum Hall effect are investigated. The statistic matrix which measures the quantity of exclusion between the quasiparticle is then evaluated. The matrix evaluated in Refs. 1 and 12 is antisymmetric but it was found to be symmetric in Ref. 13. The fractional exclusion statistics considered in these systems have two species. To my knowledge, however, the fractional exclusion statistics of the ideal gases with more than one species has not yet been investigated extensively.

As a first step, I would like to study the fractional exclusion statistics of two-species gases in any dimension in this paper.

In Sec. II the high-temperature perturbation is used to find the third virial coefficient for the two-species gases described by the Haldane exclusion statistics in arbitrary dimension. The perturbative result indicates that the two-dimensional thermodynamic potential becomes the linear function of the element of the Haldane statistics matrix, if the matrix is symmetric and the mass is species-independent. In Sec. III a nonperturbative method is used to prove this factorizable property. An exact form of the thermodynamic potential is also suggested in this section. The last section is devoted to conclusion. This paper is to generalize the investigation of the previous Letter to the system with two species.

II. VIRIAL EXPANSION

A. Formula

In the case of two species there are four elements in the Haldane statistic matrix $\alpha_{ij}$. The diagonal $\alpha_{ii}$ is the self-exclusion statistics of species $i$ while the off-diagonal $\alpha_{ij}$ ($i \neq j$) is the mutual-exclusion statistics between species $i$ and $j$. In the original definition the Haldane statistic matrix may be asymmetric. I will analyze the thermodynamics of two-species gases without assuming any symmetric property of the statistic matrix. The thermodynamic potential $Q$ in $d$-spatial dimensions for the two-species gases described by the Haldane exclusion statistics is

$$Q = -kT \int_0^{\infty} d(\varepsilon \beta) \frac{1}{(\varepsilon \beta)^{(d/2)-1}}$$

$$\times \frac{N}{\lambda_a} \ln \frac{1 + W_a}{W_a} + \frac{V}{\lambda_b} \ln \frac{1 + W_b}{W_b},$$

where $\lambda_i$ is the thermal wavelengths of the $i$ species which has mass $m_i$. The function $W_i(\varepsilon)$ is defined by the relations
\[(1 + W_a)^{-1 - a_{ab}} W_a^{a_{ab}} (W_b/W_a)^{\alpha_{ba}} = e^{(\mu - \mu_a)\beta}, \quad (2.2a)\]

\[(1 + W_b)^{-1 - a_{ab}} W_b^{a_{ab}} (W_a/W_b)^{\alpha_{ba}} = e^{(\mu - \mu_b)\beta}. \quad (2.2b)\]

The distribution function \(n_i\) can be evaluated from the equations

\[n_a = \frac{W_a + \alpha_{ab} - \alpha_{ba}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \quad (2.3a)\]

\[n_b = \frac{W_b + \alpha_{ba} - \alpha_{ab}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \quad (2.3b)\]

Note that we have assumed that both the self-exclusion and the mutual exclusion occur only between the particles in the same energy level.

**B. Perturbation**

It is known that both of the fugacities \(e^{\mu_i\beta}(= \xi_i)\) and functions \(W_i^{-1}\) can be regarded as small functions in the high-temperature limit. The method of the iterative expansion can then be adopted to find the physical quantities to any order.

First, from Eq. (2.2a) it is easy to see that \(W_a^{-1}\) can be expressed as a functional form \(W_a^{-1}[\xi_a, W_b^{-1}, \alpha_{ij}]\) through the iterative expansion to any order. Then, substituting this functional form into Eq. (2.2b) we can use the iterative expansion to express \(W_b^{-1}\) as a functional form

\[W_b^{-1} = W_a^{-1}[\xi_a, \xi_b, \alpha_{ij}]]. \quad (2.4a)\]

Interchanging the indices of \(a\) and \(b\) in Eq. (2.4a) we can then obtain the functional form of \(W_a^{-1}\)

\[W_a^{-1} = W_a^{-1}[\xi_a, \xi_b, \alpha_{ij}]. \quad (2.4b)\]

Next, substituting Eqs. (2.4a) and (2.4b) into Eqs. (2.3a) and (2.3b) we see that the distribution functions \(n_i\) can be expressed as the functional form

\[n_a = n_a[\xi_a, \xi_b, \alpha_{ij}], \quad (2.5a)\]

\[n_b = n_b[\xi_a, \xi_b, \alpha_{ij}]. \quad (2.5b)\]

Now, substituting Eq. (2.5a) into the definition

\[N_a = \int_0^\infty d(\epsilon) \beta \frac{1}{\Gamma(D/2)} (\epsilon \beta)^{d/2 - 1} V_a^{\Delta} n_a, \quad (2.6a)\]

in which \(N_a\) is particle number of \(a\) species we can, after the integration, express the fugacity \(\xi_a\) as a functional form \(\xi_a[N_a, N_b, \alpha_{ij}].\) Substituting this functional form into Eq. (2.5b) the distribution functions \(n_b\) can then be expressed as a functional form \(n_b = n_b[N_a, \xi_b, \alpha_{ij}].\)

Finally, substituting this form into the definition

\[N_b = \int_0^\infty d(\epsilon) \beta \frac{1}{\Gamma(D/2)} (\epsilon \beta)^{d/2 - 1} V_b^{\Delta} n_b, \quad (2.6b)\]

in which \(N_b\) is a particle number of \(b\) species we can again, after the integration, express the fugacity \(\xi_b\) as a functional form \(\xi_b[N_a, N_b, \alpha_{ij}].\) By interchanging the indices of \(a\) and \(b\) in the functional form \(z_b[N_a, N_b, \alpha_{ij}]\) we therefore obtain the functional form \(z_a[N_a, N_b, \alpha_{ij}].\)

Note that the functional forms mentioned in this section can be obtained through the iterative expansion to any order.

**C. Results**

The fugacity of the \(a\) species so obtained is

\[z_a = e^{\mu_a \beta} = \frac{N_a \lambda_a^d}{V} - 2 - d/2(1 - 2 \alpha_{aa})^2 \frac{N_a \lambda_a^d}{V} + 2^{-d/2}(\alpha_{aa} + \alpha_{ba}) \frac{N_a \lambda_a^d}{V} N_b \lambda_b^d \frac{N_b \lambda_b^d}{V} \]

\[+ 2^{-d/2}(1 - 2 \alpha_{aa})^2 - 3 - d/2(2 - 9 \alpha_{aa} + 9 \alpha_{aa}^2) \frac{N_a \lambda_a^d}{V} \]

\[+ 2^{-d/2}(1 - 2 \alpha_{aa})^2(1 + 2 \alpha_{ab} + \alpha_{ba} + \alpha_{aa}) - 3 - d/2(2 \alpha_{ab} + \alpha_{ba})(1 - 2 \alpha_{ab} + \alpha_{ba}) \]

\[\times \frac{N_a \lambda_a^d}{V} \frac{N_b \lambda_b^d}{V} - 2^{-d/2}(d - 3) \alpha_{ab}^2 \frac{N_a \lambda_a^d}{V} \frac{N_b \lambda_b^d}{V} + \frac{N_a \lambda_a^d}{V} \frac{N_b \lambda_b^d}{V} \]

\[+ 2(1 + d - 3d/2 - 1 + 2 \alpha_{aa}) - 3 - d/2(1 + 2 \alpha_{aa}) \alpha_{ab} \]

\[+ 2(1 + d - 3d/2 - 1 + 2 \alpha_{aa}) - 3 - d/2(1 + 2 \alpha_{aa}) \alpha_{ab} \]

\[+ \text{higher orders.} \quad (2.7)\]

The fugacity of the \(b\) species can be found from the above relation by interchanging the indices of \(a\) and \(b\). Substituting the found form of the fugacity into the expressions \(W_i^{-1}[\xi_a, \xi_b, \alpha_{ij}]\) in Eqs. (2.4a) and (2.4b), we can then obtain from Eq. (2.1) the final result.
\[ Q = -kT \left[ N_a - 2^{-1-d/2} (1-2\alpha_{aa}) \frac{N_a \lambda_a^d}{V} N_a^2 + 2^{-d/2} \alpha_{ab} \frac{N_a N_b^d}{V} N_a N_b + \left[ 3^{-1/2} 2^{-d} - 2 \cdot 3^{-1-d/2} - (2^{-d} - 3^{-1-d/2})(\alpha_{aa} - \alpha_{bb}^2) \right] \right] \]

\[ \times \left( \frac{N_a \lambda_a^d}{V} \right)^2 N_a + \left[ 2^{-d} \alpha_{ba} \left( \frac{N_a \lambda_a^d}{V} \right)^2 + (2^{-d} - 3^{-d/2}) (-1 + 2 \alpha_{aa} + \alpha_{bb}) \alpha_{ba} \right] \left( \frac{N_a \lambda_a^d}{V} \right)^2 + \frac{N_a \lambda_a^d}{V} N_b \left( N_a^2 N_b \right) + \text{terms with } a\leftrightarrow b + \text{higher orders.} \] (2.8)

It is interesting to see that in the two-dimensional system the thermodynamic potential becomes

\[ Q = -kT \frac{V}{\lambda^2} \left[ \frac{N_a \lambda^2}{V} + \frac{N_b \lambda^2}{V} \right] \left[ \left( 1 - 2\alpha_{aa} \right) \left( \frac{N_a \lambda^2}{V} \right)^2 + \left( 1 - 2\alpha_{bb} \right) \left( \frac{N_b \lambda^2}{V} \right)^2 \right] + \frac{1}{2} \left( \alpha_{ab} + \alpha_{ba} \right) \frac{N_a \lambda^2}{V} \frac{N_b \lambda^2}{V} + \frac{1}{36} \left( \frac{N_a \lambda^2}{V} \right)^3 \]

\[ + \frac{1}{12} \left( \alpha_{ab} - \alpha_{ba} \right) \left[ 2 - 4\alpha_{aa} + \alpha_{bb} + 3 \alpha_{ba} \right] \left( \frac{N_a \lambda^2}{V} \right)^2 \frac{N_b \lambda^2}{V} - \left[ 2 - 4\alpha_{bb} + \alpha_{ba} + 3 \alpha_{ab} \right] \frac{N_a \lambda^2}{V} \left( \frac{N_b \lambda^2}{V} \right)^2 \] (2.9)

if the mass of species \( a \) is equal to that of species \( b \), i.e., \( \lambda_a = \lambda_b = \lambda \). Thus, in the two-dimensional system \( Q \) becomes a linear function of the element of the Haldane statistics matrix if the matrix is symmetric, i.e., \( \alpha_{ab} = \alpha_{ba} \).

Note that the statistics matrix investigated in Refs. 1 and 12 is antisymmetric, but that investigated by Wilczek \( ^{13} \) is symmetric. They all argued that it can be realized in the layered Hall effect or high-temperature superconductor.

### III. FACTORIZATION

As suggested by the high-temperature limit in Eq. (2.9), the thermodynamic potential in two dimensions may be a linear function of the element of the Haldane statistics matrix if the matrix is symmetric and the mass is species-independent. I will prove this property in this section without using the perturbative method.

Before proving this property we need some relations. First, by differentiating Eqs. (2.2a) and (2.2b) with respect to \( \alpha_{aa} \) we have the relations

\[ \frac{1}{W_a(1+W_a)} W_{a,aa} = \frac{-(W_b + \alpha_{bb}) \left( \beta \mu_{a,aa} + \ln \frac{W_a}{1+W_a} \right) + \beta \alpha_{ba} \mu_{b,aa}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \] (3.1a)

\[ \frac{1}{W_b(1+W_b)} W_{b,aa} = \frac{\alpha_{ba} \left( \beta \mu_{a,aa} + \ln \frac{W_a}{1+W_a} \right) - (W_a + \alpha_{aa}) \beta \mu_{b,aa}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \] (3.1b)

In the same way, differentiating Eqs. (2.2a) and (2.2b) with respect to \( \alpha_{ab} \) we have

\[ \frac{1}{W_a(1+W_a)} W_{a,ab} = \frac{\alpha_{ba} \left( \beta \mu_{b,ab} + \ln \frac{W_a}{1+W_a} \right) - (W_b + \alpha_{bb}) \beta \mu_{a,ab}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \] (3.2a)

\[ \frac{1}{W_b(1+W_b)} W_{b,ab} = \frac{-(W_a + \alpha_{aa}) \left( \beta \mu_{b,ab} + \ln \frac{W_a}{1+W_a} \right) + \beta \alpha_{ab} \mu_{a,ab}}{(W_a + \alpha_{aa})(W_b + \alpha_{bb}) - \alpha_{ab} \alpha_{ba}}. \] (3.2b)

The relations after differentiating Eqs. (2.2a) and (2.2b) with respect to energy \( \epsilon \) lead to

\[ W_{a,\epsilon} = \beta n_a W_a (1+W_a) \frac{W_b + \alpha_{bb} - \alpha_{ba}}{W_b + \alpha_{bb} - \alpha_{ab}}, \] (3.3a)

\[ W_{b,\epsilon} = \beta n_b W_b (1+W_b) \frac{W_a + \alpha_{aa} - \alpha_{ab}}{W_a + \alpha_{aa} - \alpha_{ba}}. \] (3.3b)

Now, from Eqs. (3.3a) and (3.3b) we see that if the Haldane statistics matrix is symmetric then we have two simplified relations...
Fractional exclusion statistics of two-...\(\frac{d}{d\varepsilon} \ln \frac{1 + W_a}{W_a} = -\beta n_a \quad \text{(if } \alpha_{ab} = \alpha_{ba})\).\]

\[
\frac{d}{d\varepsilon} \ln \frac{1 + W_b}{W_b} = -\beta n_b \quad \text{(if } \alpha_{ab} = \alpha_{ba})\).\]

Therefore from Eqs. (2.1), (3.1a), (3.1b), (3.3a), (3.3b), (4.4a), and (4.4b) we can find that

\[
Q_{,a_{aa}} = -kT \int_0^\infty \frac{d(\varepsilon \beta)}{\lambda^2} \left[ \ln \frac{1 + W_a}{W_a} + \ln \frac{1 + W_b}{W_b} \right] \alpha_{aa} \]

\[
= -kT \int_0^\infty \frac{d(\varepsilon \beta)}{\lambda^2} \left[ \beta n_a \mu_{a,a_a} + \beta n_b \mu_{b,a_a} - n_a \ln \frac{1 + W_a}{W_a} \right] \]

\[
= -N_a \mu_{a,a_a} - N_b \mu_{b,a_a} - \frac{1}{2} kT \int_0^\infty \frac{d(\varepsilon \beta)}{\lambda^2} \left( \ln \frac{1 + W_a}{W_a} \right)^2 \]

\[
= -N_a \mu_{a,a_a} - N_b \mu_{b,a_a} + \frac{1}{2} kT \ln \left( \frac{1 + W_a(0)}{W_a(0)} \right)^2. \quad \text{(3.5a)}
\]

if the mass of species \(a\) and \(b\) is equal, i.e., \(\lambda_a = \lambda_b = \lambda\). Interchanging the indices of \(a\) and \(b\) in the above relation we have

\[
Q_{,b_{bb}} = -N_a \mu_{a,b_b} - N_b \mu_{b,b_b} + \frac{1}{2} kT \ln \left( \frac{1 + W_b(0)}{W_b(0)} \right)^2. \quad \text{(3.5b)}
\]

In the same way, from Eqs. (2.1), (3.2a), (3.2b), (3.3a), (3.3b), (4.4a), and (4.4b) we can find

\[
Q_{,a_{ab}} = -N_a \mu_{a,a_b} - N_b \mu_{b,a_b} - kT \int_0^\infty \frac{d(\varepsilon \beta)}{\lambda^2} \ln \frac{1 + W_a}{W_a} \frac{d}{d\varepsilon} \ln \frac{1 + W_b}{W_b}, \quad \text{(3.5c)}
\]

\[
Q_{,b_{ba}} = -N_a \mu_{a,b_a} - N_b \mu_{b,b_a} - kT \int_0^\infty \frac{d(\varepsilon \beta)}{\lambda^2} \ln \frac{1 + W_b}{W_b} \frac{d}{d\varepsilon} \ln \frac{1 + W_a}{W_a}. \quad \text{(3.5d)}
\]

Now, if \(\alpha_{ab} = \alpha_{ba}\) then the last integration in Eq. (3.5c) shall be equal to that in Eq. (3.5d). Therefore we see that

\[
\text{Integration in Eq. (3.5c)=Integration in Eq. (3.5d)=} \frac{1}{2} \frac{V}{\lambda^2} \ln \left[ \frac{1 + W_a(0)}{W_a(0)} \right] \left[ \frac{1 + W_b(0)}{W_b(0)} \right]. \quad \text{(3.6)}
\]

Thus the functions \(W_i(0)\) do not depend on the statistics matrix.

Next, substituting Eqs. (3.8a) and (3.8b) into Eqs. (2.2a) and (2.2b) (letting \(\varepsilon = 0\)) we can find the explicit form of the chemical potential

\[
\beta \mu_a = \alpha_{aa} N_a \lambda^2/V + \alpha_{ab} N_b \lambda^2/V + \ln \left[ 1 - \exp(-N_a \lambda^2/V) \right], \quad \text{(3.9a)}
\]

\[
\beta \mu_b = \alpha_{bb} N_b \lambda^2/V + \alpha_{ab} N_a \lambda^2/V + \ln \left[ 1 - \exp(-N_b \lambda^2/V) \right]. \quad \text{(3.9b)}
\]

Thus we see that the chemical potential \(\mu_i\) is the linear function of the element of the Haldane statistics matrix.

These complete my proof that, if the particle mass is species-independent and the statistics matrix is symmetric, then the two-dimensional thermodynamic potential can be expressed as

\[
Q(N_a,N_b,\alpha_{ij}) = C_0 + C_{aa} \alpha_{aa} + C_{bb} \alpha_{bb} + C_{s} \alpha_{s}. \quad \text{(3.10)}
\]

where \(C_i\) does not depend on the statistics matrix.

Let me try to find the constants \(C_i\) in the above expression by the following four steps.
Step (I). Setting $\alpha_{ij} = 0$ in Eqs. (2.2a) and (2.2b) we find that $W_a = e^{(e - \mu_a)\beta} - 1$ and $W_b = e^{(e - \mu_b)\beta} - 1$. The exact form of the chemical potential $\mu_i$ can be obtained from Eqs. (3.9a) and (3.9b). Then we see that $Q = Q_0(N_a) + Q_0(N_b)$, in which $Q_i(N_i)$ is defined by Eqs. (1.2a) and (1.2b). Thus $C_{ii} = Q_0(N_i) + Q_0(N_i)$.

Step (II). Setting $\alpha_{ij} = \delta_{ia}\delta_{ib}$ in Eqs. (2.2a) and (2.2b) we find that $W_a = e^{(e - \mu_a)\beta} - 1$ and $W_b = e^{(e - \mu_b)\beta} - 1$. The exact form of the chemical potential $\mu_i$ can be obtained from Eqs. (3.9a) and (3.9b). Then we see that $Q = Q_1(N_a) + Q_0(N_b)$ and thus $C_{ii} = Q_1(N_i) - Q_0(N_i)$.

Step (III). Setting $\alpha_{ij} = \delta_{ib}\delta_{ja}$ in Eqs. (2.2a) and (2.2b) we find that $W_a = e^{(e - \mu_a)\beta} - 1$ and $W_b = e^{(e - \mu_b)\beta} - 1$. The exact form of the chemical potential $\mu_i$ can be obtained from Eqs. (3.9a) and (3.9b). Then we see that $Q = Q_1(N_a) + Q_0(N_b)$ and thus $C_{ii} = Q_1(N_i) - Q_0(N_i)$.

Collecting the above results I finally find that

$$Q(N_a, N_b, \alpha_{ij}) = Q_0(N_a) + Q_0(N_b) + [Q_1(N_a) - Q_0(N_a)]C_{aa} + [Q_1(N_b) - Q_0(N_b)]C_{bb} - kTN_a N_b \frac{\lambda^2}{V} \alpha_s,$$

$$= - (N_a kT + N_b kT) + kT \frac{V}{\lambda^2} \left( 1 - 2 \alpha_{aa} \right) \left( \frac{N_a \lambda^2}{V} \right)^2 + \left( 1 - 2 \alpha_{bb} \right) \left( \frac{N_b \lambda^2}{V} \right)^2 + \alpha_s \frac{N_a \lambda^2}{V} \frac{N_b \lambda^2}{V}.$$  (3.11)

These complete the investigations. It is interesting to see that only the second Virial coefficient will depend on the Haldane statistics matrix.

IV. CONCLUSION

In some physical systems the fractional exclusion statistics should be described as more than one species. To my knowledge, however, the quantum statistical mechanics of the gases with fractional exclusion statistics for the system with many species have not yet been investigated extensively.\textsuperscript{14} In this paper the high-temperature perturbation is used to find the third Virial coefficient for the two-species gases described by the Haldane exclusion statistics in arbitrary dimension. The perturbative result indicates that the two-dimensional thermodynamic potential becomes the linear function of the element of the Haldane statistics matrix, if the matrix is symmetric and particle mass is species-independent. I present a nonperturbative method to prove this factorizable property and also suggest an exact expression of the thermodynamic potential.

Finally, it is easy to see that the factorizable property also exists in any-dimensional two-species gases with a symmetric statistic matrix, if the density of state is constant in energy.\textsuperscript{15} I hope that the investigation of this paper is helpful in understanding the general property of fractional exclusion statistics of the ideal gases with many species.

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