Pricing Contingent Claims Using the Heath-Jarrow-Morton Term Structure Model and Time-Changed Lévy Processes

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Received 8 November 2011; Received in revised form 13 July 2012; Accepted 7 December 2012

Abstract

This study examines the contingent claim valuation of risky assets in a stochastic interest rate economy using the time-changed Lévy processes model developed by Carr and Wu (2004). The proposed model adopts the approach of Heath, Jarrow and Morton’s (1992) to obtain an analytical solution of European options on risky assets and futures contracts. Furthermore, this investigation develops upper bounds for American options prices using the proposed model. The upper bounds derived in this study are not only very tight and accurate for American option pricing, but can also enhance assessment and hedging efficiency in real world markets. The asset returns obtained by the proposed model are more closely match actual market phenomena presented in the option literature because the leptokurtic and asymmetric features, interest rates and volatility are stochastic over time.

Keywords: Upper bounds for American option prices, time-changed Lévy processes, stochastic Interest rates, the HJM model

1. Introduction

Although the Black and Scholes (1973) model based on geometric Brownian motion has become the standard for contingent claims valuation, the academic literature contains two stylized facts regarding asset returns in financial markets. First, empirical asset return distribution appears leptokurtic, with two heavy tails rather than the contour described as the log normal distribution, while asset return distribution is also asymmetrical, with leftwards skew, rather than having the symmetrical property of Brownian motion. For instance, Cont (2001) presents a set of empirical facts about various types of financial markets. He finds that the asset return distribution has several statistical properties as asymmetry in time scale, high peaks and heavy tails, etc. Second, if the Black and Scholes model is accurate, implied volatilities should be flat. However, the empirical findings as Rubinstein (1994) and Carr and Wu (2003) indicate that the implied volatilities in the Black and Scholes model appear skewed for U.S. stock index options since the market crash of 1987. The literature terms this phenomenon the volatility smile.

One of the most advanced approaches to these problems involves applying stochastic time changes to the Lévy process in the asset pricing model. Thus, this study initially attempts to generalize the time-changed Lévy model of Carr and Wu (2004) by incorporating stochastic interest rates specified by Heath, Jarrow and Morton (1992) (hereafter HJM). The proposed model in this investigation is more consistent with the empirical results: Bakshi, Cao and Chen (1997) indicate that an option model permitting stochastic volatility, jumps and stochastic

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**DOI:10.6126/APMR.2014.19.3.04
interest rates factors better explains the observed distribution return and Scott (1997) also obtains the same empirical result as Bakshi, Cao and Chen (1997) and identifies a significant negative correlation between changes in interest rates and stock prices.

The time-changed Lévy model was first applied to asset pricing, with Geman, Madan and Yor (2001) modeling the characteristic functions of the asset prices logarithm by devising a theory that embedded the Brownian motion in the wide classes of Lévy processes. Subsequently, Carr, Geman, Madan and Yor (2003) extended the theory of Geman, Madan and Yor (2001) and further considered the Lévy model at time changes given by the integral of a mean reverting square root process used for calibrating the stochastic volatility pattern. However, the model of Carr, Geman, Madan and Yor (2003) assumed that volatility changes are independent of asset returns, indicating the two are not correlated. To overcome this drawback, Carr and Wu (2004) performed the correction by permitting arbitrary correlations between volatility changes and asset returns. Their model can not only encompass all tractable processes in the literature, including Brownian motion, compound Poisson jumps and others which permit infinite-activity jumps within any finite time interval, but can also allow diffusion or jump component to display evidence of stochastic volatility.

Additionally, since most market traded options are American, this investigation also follows the framework of Chung and Chang (2007) to generate the analytical upper and lower bounds of American options using the time-changed Lévy processes. In fact, the framework for pricing American option considered multiple state variables and hence was computationally costly. Particularly, for the model containing time-changed Lévy processes in the American style, the analytical bounds not only provide a benchmark for American option pricing but also require minimal computational time. Furthermore, analytical upper bounds also provide useful benchmark and control variables for market participants. If option holders find that a market price exceeding the upper bound is associated with overestimation they should sell the option. Conversely, if the option holder finds that upper bound equals the intrinsic value, then they should exercise the American option. Therefore, the second aim of this study is to present an accurate and efficient method for pricing American options with minimal complexity.

Previous investigations on option bounds have adopted several approaches. For example, Levy (1985) and Constantnides and Perrakis (2002) suggested second order stochastic dominance frameworks and their analytical models incorporate taxes, transaction costs and distribution-free properties. Furthermore, Ritchken (1985) derived option bounds via a single-period model using linear programming by permitting a finite number of revision opportunities in an incomplete market. Moreover, Lo (1987), Boyle and Lin (1997) further devised the upper bounds of a European call option on both single and multiple assets by utilizing the Jensen and Cauchy inequalities. Additionally, Chen and Yeh (2002) provide more general upper bounds with the inequality of Jensen for American options under stochastic interest rates, stochastic volatility and compound poisson jumps, but these upper bounds are only appropriate in cases where the interest rate exceeds the dividend yield. Chung and Chang (2007) thus generalize and tighten the upper bounds of Chen and Yeh, which can be applied when the interest rate is either higher or lower than the dividend yield. The tightness bounds of Chung and Chang converge to accurate American option prices, even given small dividend yield or the large volatility.

In contrast, this study makes several contributions to the literature on option pricing bounds and on options under the Lévy processes. First, this study adds the stochastic interest and dividend rate factors into the time-changed Lévy processes by extending the framework of Amin and Jarrow (1991) and Guo and Hung (2007), and derives an analytical solution to European options that possess these characteristics. Compared to the model of Carr and Wu (2004), the proposed model considers two more stochastic factors: interest and dividend rates. Second, this study generates the analytical upper and lower bounds of American and futures.
options under the time-changed Lévy processes using the methods of Chung and Chang (2007), which can be applied to cases where the interest rate is higher or lower than the dividend yield and the upper bounds for American options calculated in this study broaden the applicability from a finite number of large jumps to an infinite number of small jumps within a finite time interval. Third, for both European and American options, this study provides analytical solutions via the Fast Fourier Transform (hereafter FFT) method, which is very fast and requires minimal computation.

The remainder of this paper is organized as follows. Section 2 presents fundamental analyses of the time-changed Lévy processes and an analytical solution for European options. In the option bounds theorem, the prices of European options are actually the lower bounds of the prices of American options when the stock dividend yields are positive. Section 3 presents the cases of the upper bounds of American options and futures options for the situations where interest rates are both higher than and lower than stock dividend yields. Next, section 4 derives different jump components of the time-changed Lévy processes under the risk neutral measure. Section 5 then presents numerical results for different Lévy processes. Finally, Section 6 gives conclusions.

2. Time-changed Lévy processes with Stochastic interest rates

This subsection first discusses the general properties and theorems related to the model in which the asset price is driven by the exponential of the time-changed Lévy process. Second, this subsection derives an extended model under time-changed Lévy processes with stochastic interest rates specified by the HJM model.

2.1 Review the general properties under time-changed Lévy processes

Consider a probability space \((\Omega, \mathcal{F}, P)\) with filtration \((\mathcal{F}_t)\), and an \(R^d\) valued stochastic process denoted as \(X = \{X(\tau) | \tau \geq 0\}\), called a Lévy process, which satisfies the following conditions:

(i). \(X_\tau\) is adapted to \(\mathcal{F}_\tau\),

(ii). \(X_\tau\) starts with a value of zero and has independent and stationary increments such that the distribution of an increment over \(\{X_{\tau+\tau'} - X_\tau : \tau \geq 0\}\) does not depend on \(\tau\),

(iii). \(X_{\tau+\tau'} - X_\tau\) has an infinitely divisible distribution and \(\phi(u)^{\tau'}\) as its characteristic function,

(iv). \(X_\tau(w)\) is right continuous with left limits, and \(\Omega_0 \in \mathcal{F}\) with \(P(\Omega_0) = 1\) such that for all \(w \in \Omega_0\).

Theorem 1: Lévy Khintchine Theorem

Let the \(d\) dimensional stochastic process \(X\) be a Lévy process exists the triplet of predictable characteristics such as the real-valued vector \(\mu\) in \(R^d\), the positive semi-definite matrix \(\Sigma\) on \(R^{d \times d}\), and the Lévy measure \(\pi\) in \(R^d\) satisfying \(\pi(\{0\}) = 0\) with \(\int (x^2 \wedge 1)\pi(dx) < \infty\). The characteristic exponent of \(X\), called the cumulant characteristic function \(\psi(u) = \log \phi(u)\), satisfies the following Lévy Khintchine formula:

\[
\psi(u) = -i\mu^T u + \frac{1}{2} u^T \Sigma u + \int_{\mathbb{R}} (1 - e^{iu^T x} + iu^T x 1_{|u| > 1})\pi(x)dx,
\]

and the characteristic function of \(X\)

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\[ \phi_{x_t}(u) \equiv E_x[e^{iuX_t}] = e^{-\tau \psi_x(u)}. \]  

(2)

Based on Theorem 1 and considering the \(d\) dimensional random process \(Y\) by estimating \(X\) at \(t\), under the risk neutral measure \(Q\), the generalized Fourier transform of the time-changed Lévy process is expressed by

\[ \phi_{y_t}(u) \equiv E_y[e^{iuY_t}] = E_y[e^{-\tau \psi_y(u)}] = \mathcal{L}_y(\psi_y(u)). \]  

(3)

The above time-changed Lévy process \(Y_t\) corresponds to \(X_{T_t}\) where the subscript \(T_t\) denotes a random time change in a Lévy process. Here, \(\tau\) is conceived as the time during the calendar date and \(T_t\) as the time during the trading day for a real-world securities market. In practice, Eq.(3) enables the problem of acquiring the generalized Fourier transform of securities return to be simplified to price the Laplace transform of random time under a measure of complex value. Thus, Theorem 1 of Carr and Wu (2004) can be used to define the new complex valued measure \(M\) with respect to the risk neutral measure \(Q\) as

\[ \frac{dM}{dQ} \equiv \exp(iu'Y_t + T_t\psi_y(u)). \]  

(4)

In terms of Eq.(4), under the predetermined interest rate setting, the generalized Fourier transform of security returns under new measure \(M\) can be specified as follows:

\[ \phi_{v_T}(u) \equiv E_v[e^{iu(\ln(T/1))}] = e^{iu(r-q)\tau} \cdot E_v[e^{iu(w-\frac{1}{2})\tau + iu(J_{T-T_T})}] \]

\[ = E_v[\exp(iu(r-q)\tau - \tau \psi_x - \tau \psi_y)] \]

\[ = e^{iu(r-q)\tau} \cdot E_v[e^{-\psi_y(T_t)}] \]

\[ = e^{iu(r-q)\tau} \cdot \mathcal{L}_v[e^{-\psi_y(T_t)}] \]  

(5)

where \(\tau\) denotes the period \([0, T]\), \(\sigma W_t - \frac{1}{2} \sigma T_t\) is the Brownian motion term representing the diffusion component, \(J_{T-T_T}\) is the composition of the jump component, \(\psi = [\psi_x, \psi_y]\) is the vector of the characteristic exponents denoting the diffusion and jump respectively, and \(\xi\) is the analogous concavity adjustment for \(J\) based on Huang and Wu (2004).

Intuitively, Eq.(5) is combined the Laplace transformation of two characteristic exponents from the diffusion and jump components with the difference between the interest rate and dividend yield. Furthermore, the random time \(T_t\) is continuous and differentiable with respect to \(\tau\). Let

\[ \nu(\tau) \equiv \begin{bmatrix} \nu'((\tau), \nu'((\tau)) = \frac{\partial T_t}{\partial \tau}. \]  

(6)

Then \(\nu'((\tau)\) is propositional to the instantaneous activity rate of the diffusion component and \(\nu'((\tau)\) is proportional to instantaneous activity rate of the jump component. Using the assumption of Carr and Wu’s (2004), the instantaneous activity rate satisfies the following mean reverting square root process

\[ dv(t) = \alpha(\beta - v(t))dt + \sigma \sqrt{v(t)}dW_t(t) \]  

(7)
where $\alpha$, $\beta$ and $\sigma$, are parameters in the mean reverting square root process, and $dW(t)$ is defined as the standard Brownian motion and correlates with the stock return process as
\[ E^o [dW_s(t)dW_v(t)] = \rho_{sv} dt. \] (8)

Huang and Wu (2004) indicated that applying separate random time changes to different Lévy components, a single activity rate can be embedded in the behavior of the intensity of diffusion, thus generating normal return innovation, while the other activity rate controls the jump component, thus yielding negatively skewed behavior. To capture stochastic volatilities in security returns, the analytical approach presented here incorporates two different activity rate processes specifically representing the diffusion and jump components into our analysis.

2.2 The model

This study extends the Amin and Jarrow’s model under the stochastic interest and dividend rate specifications and also employs time-changed Lévy processes to calculate the asset price returns. This model considers all stylized evidences for the equity index returns, which presented in the empirical studies of Cont (2001) and Wu (2006). Therefore, the model objective is to correspond as closely as possible to the tail behavior of asset return innovations. Notably, this study uses the following mathematical notations.

$$f(r, T):$$ The forward interest rate at time $t$ with maturity $T$.

$$f_q(t, T):$$ The forward dividend rate at time $t$ with maturity $T$.

$$P_r(t, T):$$ The price at time $t$ of zero coupon bond with interest rate maturity $T$.

$$P_q(t, T):$$ The time $t$ price of zero coupon bond with dividend rate maturity $T$.

$$S(t):$$ The price at time $t$ of equity.

$$r(t):$$ The interest rate at the time $t$ and $r(t) = f_r(t, t)$.

$$q(t):$$ The dividend rate at the time $t$ and $q(t) = f_q(t, t)$.

$$F(t, T):$$ Time $t$ price of futures contracts, with term to maturity $T$.

$$B_r(t, T) = \exp\left[\int_0^t r(u) du\right]$$, the time $t$ price of the money market account in terms of forward interest rate with initial value $B_r(0, 0) = 1$.

$$B_q(t, T) = \exp\left[\int_0^t q(u) du\right]$$, the time $t$ price of the money market account in terms of forward dividend rate with initial value $B_q(0, 0) = 1$.

By incorporating the equity price dynamics into the model, Proposition 1 gives the extended Amin and Jarrow’s model under objective measure $P$ containing time-changed Lévy processes.

**Proposition 1:** Under the domestic objective measure $P$ for any $t \in [0, T]$, the dynamics of the stock price, forward interest rate, and dividend rate are

$$\frac{dS(t)}{S(t)} = \mu_s(t) dt + \sigma_s dW_s(t) + \sigma_v dW_v(t) + \sigma_s \sigma_v \rho_{sv} dt.$$ 

$$df_r(t, T) = \alpha_r(t, T) dt + \sigma_r(t, T) dW_r(t),$$

$$df_q(t, T) = \alpha_q(t, T) dt + \sigma_q(t, T) dW_q(t), \quad 0 \leq t \leq T,$$ (9)
where \( dW_z(t)dt, dW_y(t)dt, dW_z(t)dt, dW_y(t)dt = \rho_{z,v} dt, \)
\( dW_z(t)dt, dW_y(t)dt = \rho_{z,q} dt, \)
\( dW_z(t)dt = 0, dW_y(t)dt = 0, \) and where \( \pi_x \) represents one of the density functions of time-changed Levy processes detailed in Section 2. \( \delta_{sr} \) or \( \delta_{sq} \) establishes the correlation between the stock price and the interest rate or between the stock price and the dividend rate. With the economic perspective, in general, there is rather low correlation between the volatility of individual stock and the market interest rate or the dividend rate. Hereafter, our study postulates the assumption that \( dW_z(t)dt, dW_y(t)dt = 0, \) and where \( X \) represents one of the density functions of time-changed Levy processes detailed in Section 2.

Additionally, from the empirical results of Scott (1997), equity returns are negatively correlated with volatility and interest rates, and these negative correlations significantly influence option valuation. Consequently, to reflect market reality, the proposed model incorporates the correlations, namely the \( \rho_{z,v} \) and \( \rho_{z,q} \) coefficients, between variance shocks and equity prices and those between the forward interest and dividend rate dynamics, and the correlation between interest or dividend rate and equity price is also enabled by the \( \rho_{s,v} \) and \( \rho_{s,q} \). Using \( \delta_{sr} \) and \( \delta_{sq} \), the relationship between interest or dividend rate process and asset price can be calculated as follows

\[
\frac{dS(t)}{S(t)} \cdot df(t, T) = \sigma_r(t, T) \cdot \left( \delta_{sr} + \delta_{sq} \rho_{vr} + \sigma_r \rho_{vr} \right) dt, \tag{10}
\]

To convert the objective measure \( P \) into the risk neutral measure \( Q \), it is necessary to specify the conditions of the stock price and forward rate dynamics such that a unique equivalent martingale probability measure exists. These conditions are given by the following Lemma 1:

**Lemma 1: Arbitrage-free term structure**

\[
\frac{P(t, T)}{B(t, T)} \text{ and } \frac{S(t)B_{q}(t)}{B(t, T)} \text{ are } Q \text{ martingales if and only if the following conditions are satisfied}
\]

\[
\alpha_r(t, T) = \sigma_r(t, T) \left[ \int_0^T \sigma_r(t, u) du - \lambda_r(t) \right] \]

\[
\alpha_q(t, T) = \sigma_q(t, T) \left[ \left( \int_0^T \sigma_q(t, u) du \right) dt - \lambda_q(t) - \delta_{s,q} \rho_{rq} - \delta_{s,q} \lambda_q(t) - \sigma_q \lambda_q(t) \right] \]

\[
\mu_r(t) = r(t) - q(t) - \delta_{s,q} \lambda_q(t) - \delta_{s,q} \lambda_q(t) - \sigma_q \lambda_q(t) \tag{11}
\]

**Proof:** see Appendix A

**Proposition 2:** Under the risk neutral measure \( Q \), for any \( t \in [0, T] \), the dynamics of the pure discount bond price of interest rate and stock price are

\[
\frac{dS(t)}{S(t)} = \left[ r(t) - q(t) \right] dt + \delta_{sr} dW^v_r(t) + \delta_{sq} dW^v_q(t) + \sigma_r dW^v_z(t) + \left( e^{\omega} - 1 \right) d\pi^v_z(t) - E^v \left[ \left( e^{\omega} - 1 \right) d\pi^v_z(t) \right].
\]
\[
\frac{dP(t,T)}{P(t,T)} = r(t)dt + a_{r}(t,T)dW_{r}^{o}(t),
\]
\[
\frac{dP(t,T)}{P(t,T)} = \left( q(t) - \left( \delta_{q} + \rho_{q} + \rho_{q}\sigma_{x} \right) a_{q}(t,T) dt \right) + a_{q}(t,T)dW_{q}^{r}(t) \quad 0 \leq t \leq T, \tag{12}
\]
where \( a_{q}(t,T) \equiv -\int_{0}^{T} \sigma_{r}(t,u)du \) and \( i \in \{ r, q \} \).

Proof: refer to Appendix A.

Notably, as in Proposition 2, for the forward dividend rate dynamic, the drift of the forward interest rate under risk neutral measure \( Q \), remain unchanged and the zero coupon bond prices in terms of interest rate still follow the log normal distribution. However, for the forward dividend rate dynamic, the drift appears to continue for one more term, \( \left( \delta_{a} + \rho_{a} + \rho_{a}\sigma_{x} \right) a_{a}(t,T) \), producing the interactive correlations between the equity price, forward interest rate and forward dividend rate.

To price derivatives in a stochastic interest rate environment, the risk neutral measure \( Q \) must be transformed into forward neutral measure \( TQ \) such that the relative price of \( S(t) \) associated with the zero coupon bond price, \( \frac{S(t)}{P(t,T)} \), is a martingale.

**Proposition 3:** Under the forward neutral measure \( TQ \) for any \( 0 \leq t \leq T \leq T_{1} \), the dynamics of the forward stock price and forward zero coupon bond price of interest rate are expressed as follows:

\[
\frac{dS_{r}^{T}(t,T_{1})}{S_{r}^{T}(t,T_{1})} = \left[ \delta_{r} - a_{r}(t,T_{1}) \right] dW_{r}^{o}(t) + \left[ \delta_{s} + a_{s}(t,T_{1}) \right] dW_{s}^{r}(t) + \sqrt{\nu_{s}} dW_{s}^{q}(t)
\]
\[
+ \left( e^{\nu_{s}} - 1 \right) d\pi_{s}^{\nu_{s}}(t) - E^{\nu_{s}} \left[ \left( e^{\nu_{s}} - 1 \right) d\pi_{s}^{\nu_{s}}(t) \right],
\]
\[
\frac{d\left( P_{r}^{T}(t,T_{1}) \right)}{P_{r}^{T}(t,T_{1})} = \left( a_{r}(t,T_{1}) - a_{r}(t,T_{1}) \right) dW_{r}^{o}(t)
\]
where \( P_{r}^{T}(t,T_{1}) \) and \( S_{r}^{T}(t,T_{1}) \) individually represent the time \( t \) forward prices defined by \( P_{r}(t,T) \), where \( 0 \leq t \leq T \leq T_{1} \).

Proof: see Appendix B.

The above equations can develop the characteristic function of the stock price. Finally, for pricing contingent claims under forward neutral measure, transform the relative price of this asset under the risk neutral measure associated with the forward neutral measure is necessary. Hence, the time 0 price of a European option with maturity \( T \) is

\[
E^{Q}_{0} \left[ \frac{1}{B_{r}(0,T)} \max \left[ S(T) - K, 0 \right] \right] = P_{r}(0,T)E^{Q}_{0} \left[ \max \left[ \frac{S(T)P_{r}(T,T)}{P_{r}(T,T)} - K, 0 \right] \right] \tag{14}
\]
where \( E^{Q}_{0} \) and \( E^{Q}_{0} \) are the conditional expectations at time 0 under the risk neutral measure \( Q \) and forward neutral \( Q^{T} \) respectively.
2.3 The valuation of futures contracts and associated options

Cont (2001) empirically examined the returns density of S&P 500 index futures over 30 minute intervals and identified high peaks and heavy tails. To represent the above characteristics in the returns of futures contracts of risky assets, this study incorporates the time-changed Lévy processes into assessing the European futures options in an economy with stochastic interest rates. Actually, a risky asset paying a stochastic dividend yield can be considered identical to the spot exchange rate paying the risk less rate of foreign country.

Let \( E[\cdot | \mathcal{F}_t] \) denote the conditional expectation under an equivalent martingale measure conditional on the information at date \( t \). The futures price must be equivalent to the expectations of the spot price at maturity \( T \), and thus the time \( t \) of the futures price is given by

\[
F(t,T) = E_t [ F(T,T) | \mathcal{F}_t ] = E_t [ S(T) | \mathcal{F}_t ]
\]

\[
= S(t) e^{\int_{(f_T,f_0)} - r_f(s) ds}
\]

\[
= \frac{S(t) P_q(t,T)}{P_r(t,T)}
\]

(15)

for any given date \( 0 \leq t \leq T \leq T_1 \).

The price of a European call option with maturity date \( T \) and strike price \( K \) on the futures contract with maturity \( T_1 > T \), \( C^r \), is given by

\[
C^r = E^q_0 \left[ \frac{1}{B(T,T)} \max \left( F(T,T) - K, 0 \right) \right]
\]

\[
= P_r(0,T) E^q_r \left[ \max \left( \frac{F(T,T)}{P(T,T)} - \frac{K}{P_r(T,T)}, 0 \right) \right].
\]

(16)

Similar to the above statement, to evaluate European futures options in a stochastic interest rate environment, Eq.(16) must be transformed from risk neutral measure \( Q \), into a forward neutral measure \( Q^T \) such that the relative price of \( F(T,T_1) \) associated with the zero coupon bond price, \( F(T,T_1)/P(T,T) \), is a martingale under the forward neutral measure.

The new futures price dynamics considering the forward neutral measure price are then given by the following Proposition 4.

**Proposition 4:** Under the forward neutral measure \( Q^T \) for any \( 0 \leq t \leq T \leq T_1 \), the futures price dynamic is as follows:

\[
F(T,T_1) = \frac{F(T,T)}{P_r(T,T)} \exp \left[ \int_0^T \left( \delta_{S_T} (v) - a_r(v,T_1) - a_r(v,T) \right) dW_r^{Q^r} (v) \right]
\]

\[
+ \left( \delta_{S_T} (v) + a_q(v,T_1) \right) dW_q^{Q^r} (v) + \sigma_S dW_S^{Q^r} (v) + \left( e^{r_T (v)} - 1 \right) d\pi_S^{Q^r} (v)
\]

\[- E^{Q^r} \left[ \left( e^{r_T (v)} - 1 \right) d\pi_S^{Q^r} (v) \right] + \frac{F(T,T_1)}{P_r(0,T)} \exp \left[ - \frac{1}{2} \int_0^T \left( \delta_{S_T} (v) - a_r(v,T_1) - a_r(v,T) \right) \right]
\]

\[
+ \left( \delta_{S_T} (v) + a_q(v,T_1) \right) + \sigma_S \right]^2 dv \]

(17)
3. Upper bounds for American and futures calls on dividend paying stocks

This Section introduces the upper bounds for American and futures call options on dividend paying stocks. In fact, the upper bounds for American call options equal those for American futures options. Therefore, the following subsection only introduces the optimal method of obtaining the upper bounds for American options. The concept of upper bounds resembles a binomial method for American options to compare the risk neutral expectations of discounted payoff function with the intrinsic value of an option. If one discounted payoff function always equals or exceeds the intrinsic value at any given time, then American call options remain alive until maturity, and this function must represent the upper bound for American call options.

This idea first originated from Chen and Yeh (2002), but Chung and Chang (2007) calculated much tighter upper bounds than previous studies. This investigation applies the approach of Chung and Chang to the proposed model, in the situations where the interest rate both exceeds and is smaller than the dividend yield. Next, this study interprets the prerequisites for the generalized theorems employed in the following subsections.

**Theorem 2:**

Let $T$ denote the expiration of the state contingent option claim. Furthermore, define $Y(t,T) = h(t,T)G(T)$, where $G(t)$ $G(t)$ represents the intrinsic value of an option at time $t$, and $h(t,T)$ is any function which satisfies:

(a). $h(t,s) \geq \delta(t,s)$ for any $t < s$, where $\delta(t,s)$ is the discount factor from time $t$ to $s$,

(b). $h(t,T) = h(t,s)h(s,T)$ for any $s \in (t,T)$,

(c). $h(t,t) = 1$.

If $E(Y(t,T)) > G(t)$ for all $t$, then $E(Y(t,T))$ represents the upper bound of the American option value at time $t$. The proof indicates the generalized theorem 1 of Chung and Chang (2007).

Comparing Chen and Yeh’s (2002) Theorem 1 with Theorem 1 of Chung and Chang (2007), the upper bound of the former represents a special case in the setting of the latter when the function $h(t,T)$ equals exactly one, but the appropriate function $h(t,T)$ is always less than one. Consequently, the upper bound of Chung and Chang (2007) is tighter than that of Chen and Yeh (2002). Theorem 2 demonstrates that the discounted processes of upper bounds are actually super martingale processes, as follows:

$$E(Y(t,T)) \geq E(\delta(t,s)E(Y(s,T)))$$

$$E(G(T)) \geq E(\delta(t,s)E(G(T)))$$

where the variety of discounted time is $t < s < T$. Furthermore, Chung and Chang (2007) further develop above Theorem 2, such that $Y(t,T)$ can be any random variables which satisfies similar principles. This study constructs Theorem 3, as follows:
Theorem 3:

Let $T$ represent the maturity of the option contract. Furthermore, $Y(t,T)$ is a random variable at time $t$, which satisfies

a. $Y(T,T) \geq G(T)$,

b. $E(Y(t,T)) \geq E(\tilde{\sigma}(t,t+\Delta t)Y(t + \Delta t,T))$ for any $t \in [0,T - \Delta t]$, 

c. $E(Y(t,T)) \geq G(t)$, for any $t \in [0,T]$,

where $G(t)$ is the intrinsic value of the option at time $t$, and $E(Y(t,T))$ represents an upper bound of an American call option. The proof involves the generalized theorem 2 of Chung and Chang (2007).

The above conditions have several intuitive implications. First, any upper bound must satisfy the boundary condition. Second, the current upper bound always exceeds the discount value for the upper bound before the maturity date. Third, the upper bound must always exceed the intrinsic value of the option. Satisfying all three conditions demonstrates that $E(Y(t,T))$ is an upper bound of an American call option.

3.1 American calls and future calls on dividend paying stocks when $r > q$

Theorem 2 shows that a suitable $Y(t,T)$ can be identified that exceeds the intrinsic value at any time before maturity $T$, then $E(Y(t,T))$ is an upper bound of the American option value at time $t$. Substituting Theorems 2 and 3 into the proposed model, $Y(0,T) = e^{\int_0^T (\alpha + \gamma(t))dt} \max(S(T) - K,0)$ can be selected as an upper bound of the American option price, and furthermore it can be found that $E_e(Y(0,T)) \geq G(0)$, so that for any $t \in [0,T]$,

$$E_e\left[e^{\int_0^T (\alpha + \gamma(t))dt} \max(S(T) - K,0)\right] \geq \max \left[ E_e\left[S(0)e^{\int_0^T (\alpha + \gamma(u))du} - Ke^{\int_0^T (\alpha + \gamma(u))du}\right], 0\right]$$

$$> \max\left[S(0) - K,0\right]. \quad (19)$$

The first inequality in Eq.(19) derives from the inequality of Jensen, while the second one derives from a risk neutral measure. Therefore, the value of the upper bound of the American call option is represented as follows:

$$U_e = E_e\left[e^{\int_0^T -ydtd} \max(S(T) - K,0)\right]$$

$$= \max \left[ E_e\left[S(T)e^{\int_0^T (\alpha + \gamma(u))du}\right] - KE_e\left[e^{\int_0^T (\alpha + \gamma(u))du}\right]\right]$$

$$> \max\left[S(0) - K,0\right]. \quad (20)$$

Finally, to evaluate the time 0 price of the upper bound above in stochastic interest rate specification, Eq.(20) must be further transformed into the forward neutral measure $Q^T$, using the relative ratio of the zero coupon bond price for interest rate to the zero coupon bond price for the dividend rate numeraire, as follows:

$$U^C = E_0^Q\left[\frac{B_q(0,T) \max(S(T) - K,0)}{B_q(0,T)}\right]$$
\[
P_r(0,T) = \frac{P_q(0,T)}{P_q(0,T)} E_0^Q \left[ \max(S(T) - K, 0) P_r(T, T) \right]
\]
\[
= \frac{P_r(0,T)}{P_q(0,T)} E_0^Q \left[ \max(S_r(T, T) - K, 0) \right]
\] (21)

3.2 American and future calls on dividend paying stocks when \( r < q \)

Before undertaking the case where the interest rate is smaller than the dividend yield for an American call option, its upper bound can be obtained by applying Theorem 2 from Chung and Chang (2007). First, \( Y(0, T) = \max \left[ S(T) e^{\int_0^T (q(u) - r(u)) du} - K, 0 \right] \) is once again designated as an upper bound, where \( Y(T, T) = \max \left[ S(T) - K, 0 \right] = G(T) \) satisfies condition \( a \) of Theorem 2 of Chung and Chang. Therefore, \( Y(t, T) \) can be demonstrated to satisfy condition \( b \) of their Theorem 2, as follows:

\[
E_u[Y(0, T)] = E_u \left[ \max \left[ S(T) e^{\int_0^T (q(u) - r(u)) du} - K, 0 \right] \right]
\]

\[
> E_u \left[ \max \left[ S(T) e^{\int_{0,\Delta t}^T (q(u) - r(u)) du} - K, 0 \right] \right]
\]

\[
> E_u \left[ \max \left\{ \delta(0, 0 + \Delta t) \left[ S(T) e^{\int_{0,\Delta t}^T (q(u) - r(u)) du} - K \right], 0 \right\} \right]
\]

\[
= E_u \left[ \delta(0, 0 + \Delta t) Y(0 + \Delta t, T) \right].
\] (22)

Using Jensen’s inequality again, the last condition of Theorem 2 is calculated as follows:

\[
E_u \left[ \max \left[ S(T) e^{\int_0^T (q(u) - r(u)) du} - K, 0 \right] \right] \geq \max \left[ E_u \left[ S(T) e^{\int_0^T (q(u) - r(u)) du} - K \right], 0 \right]
\]

\[
= \max \left[ S(0) - K, 0 \right].
\] (23)

As the procedure in which the interest rate exceeds the dividend rate, Eq.(23) must be further transformed into the forward neutral measure \( Q^T \), using the relative ratio of the zero coupon bond price for interest rate to the zero coupon bond price for the dividend rate numeraire, as follows:

\[
U^C = E_0^Q \left[ \max \left[ S(T) e^{\int_0^T (q(u) - r(u)) du} - K, 0 \right] \right]
\]

\[
= e^{\int_0^T (q_u - r_u) du} E_0^Q \left[ \max \left[ S(T) - K \cdot e^{\int_0^T (q(u) - q(u)) du} \right], 0 \right]
\]
4. The valuation of contingent claims using the extended time-changed Lévy processes model

The extended Amin and Jarrow’s model with random time changes presented here employ several different independent stochastic processes. Representative examples of Lévy processes include the Gaussian process, the Compound Poisson process (CP), the Double Exponential process (Kou) and the special case of the $\alpha$ stable processes, such as the Variance Gamma process (VG), the Log Stable process (LS) and others. In the context of stochastic interest rate specification, under the forward neutral measure, the generalized Fourier transform (the characteristic function) of security return in the proposed model can be expressed by

$$
\phi_{q_0} \equiv E^{\psi}_0 \left[ e^{i \nu \ln S(t)/S(0)} \right] = S(0) \cdot E^{\psi}_0 \left[ \exp \left( -\tau \psi \right) \right] \\
= S(0) \cdot E^{\psi}_0 \left[ e^{-\psi T,} \right] \\
= S(0) \cdot L^{\psi}_c (\psi). \tag{25}
$$

where $L^{\psi}_c (\psi)$ denotes the Laplace transform of $T_c$ and the characteristic exponent $\psi$ can be derived using Lévy Khintchine formula. The random time is given by the integral

$$
T_r = \int_0^T v(s) ds,
$$

which is determined by specifying the activity rate process $v(t)$. From Eq.(25), the original forward neutral measure is transformed into the new complex value measure $M$ from Eq.(4) via the Laplace transform and then an analytical solution of the Laplace transform is obtained using the method of Carr and Wu (2004):

$$
L^{\psi}_c (\psi) = \exp (-c(\tau) - d(\tau) v(\tau)). \tag{26}
$$

For the specific one factor case, the coefficients $[c(\tau), d(\tau)]$ are

$$
c(\tau) = \frac{\psi(1-e^{-\eta \tau})}{2 \eta - (\eta - \kappa^\omega)(1-e^{-\eta \tau})};
$$

$$
d(\tau) = \frac{\alpha \beta}{\sigma_i}; 2 \ln \left( \frac{\eta - \kappa^\omega}{2 \eta - (1-e^{-\eta \tau})} \right) + (\eta - \kappa^\omega) \tau,
$$

where

$$
\eta = \sqrt{(\kappa^\omega)^2 + 2 \sigma_i^2 \psi}, \quad \kappa^\omega = \alpha \beta - i \mu \sigma_i.
$$

with the boundary conditions $c(0) = 0$ and $d(0) = 0$. Using Eq.(26), this study then derives the price of the European style state contingent claim via the FFT method. In conclusion, this study summarizes the eventual result of the characteristic diffusion exponent and characteristic jump exponents for different Lévy components under the risk neutral measure listed in Table 1. By combining the characteristic jump exponents based on different Lévy densities with
characteristic diffusion exponents in Appendix B and C, respectively, this study obtains the characteristic functions of European options and American upper bounds in the time-changed Lévy process with various cases where \( r > q, \) \( r < q, \) and futures call option.

### Table 1. The characteristic jump exponents of the Lévy measures

<table>
<thead>
<tr>
<th>Lévy components</th>
<th>Lévy measure ( \pi(dx)/dx )</th>
<th>Characteristic exponents ( \psi(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion</td>
<td>( 0 )</td>
<td>( \frac{1}{2} \sigma^2(iu + u^2) )</td>
</tr>
<tr>
<td>CP Jump</td>
<td>( \lambda_j = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp(-\frac{(x-\mu_j)^2}{2\sigma_j^2}) )</td>
<td>( \lambda_j \left[ iu(e^{\mu_j+i\sigma_j^2/2} - 1) - (e^{iu\mu_j-i\sigma_j^2/2} - 1) \right] )</td>
</tr>
<tr>
<td>VG Jump</td>
<td>( C \frac{e^{-C</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>( C \frac{e^{-M</td>
<td>x</td>
</tr>
<tr>
<td>LS Jump</td>
<td>( \frac{\sigma_j}{\sigma_j^2 + (x-\mu_j)^2} )</td>
<td>( \lambda_j (iu - (iu)^2) )</td>
</tr>
</tbody>
</table>

### 5. Numerical results

This section summarizes the numerical results for the upper and lower bounds of American calls and American futures calls with the CP, the VG and LS processes. In practice, applying stochastic time changes to the CP process is similar to incorporating stochastic volatility with simultaneous jumps into the model of Bakshi et al. (1997), but the proposed model incorporating the CP process not only encompasses all factors in the above model, but also includes a stochastic interest rates factor. The parameters of the numerical results are based on Guo and Hung (2007) and \( S(0) = 100, v(0) = 0.04, r(0) = 0.05, q(0) = 0.03, \alpha = 1.5, \beta = 1, \gamma_1 = 0.005, \gamma_2 = 0.003, k_r = 0.03, k_q = 0.04, \delta_r = 0.05, \delta_q = 0.05, \tau = 1, T = 1.5, \lambda = 0.6, \mu = 0.01, \mu_1 = 1.01(\text{LS}), \sigma_r = 0.3, \sigma_q = 0.01, \rho_{r,q} = -0.4, \rho_{r,q} = -0.05, \rho_{s,q} = 0.05, \rho_{s,q} = 0.05. \) Table 2 lists the results for the upper and lower bounds of American calls with each Lévy process when \( r > q. \) The second rows with different exercise prices in Table 2 represent the lower bounds of the American calls, which approximate those of the European calls. The cases of the three Lévy processes (namely CP, VG and LS) when \( r > q \) in Table 2 show that if the call option values are in or at the money, the call option values gradually increase whenever the correlation between stock price \( S(t) \) and activity rate \( v(t) \) increases in all four processes. This increase results from the intrinsic value, since the positive intrinsic value represents the positive impact of the stock return volatility resulting from increased correlation coefficient. However, this situation is the opposite of that in which the call option values are out of the money. The right side of Table 2 also illustrates the relationship between jump intensity measure \( \lambda \) and exercise price when \( r > q. \) Most relationships between the exercise price and jump intensity measure \( \lambda \) exhibit a consistently positive direction in Lévy processes, and a phenomenon that occurs because jump intensity measure \( \lambda \) is the mean of the jump variation degree and a larger jump variation degree increases the option volatility and price.
Table 2. Our upper bounds for American calls on dividend-paying stocks when \( r > q \)

<table>
<thead>
<tr>
<th>( \rho_n )</th>
<th>Processes</th>
<th>( K )</th>
<th>( \lambda )</th>
<th>Processes</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>Upper</td>
<td>CP</td>
<td>15.2820</td>
<td>8.9801</td>
<td>4.7889</td>
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<td>0</td>
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<td>15.2851</td>
<td>8.9788</td>
<td>4.7825</td>
<td>0.4</td>
</tr>
<tr>
<td>0.4</td>
<td>CP</td>
<td>15.2882</td>
<td>8.9776</td>
<td>4.7762</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Note. The top and bottom numbers of every process are American and European call option values, respectively.

The parameters are \( S(0)=100, \nu(0)=0.04, r(0)=0.05, q(0)=0.03, \alpha=1.5, \beta=1, \gamma=0.005, \gamma_{\alpha}=0.003, \kappa=0.03, \kappa_{\alpha}=0.04, \delta_{\alpha}=0.05, \delta_{\gamma}=0.05, \tau=1, T_{\alpha}=1.5, \lambda_{\alpha}=0.6, \mu_{\lambda_{\alpha}}=0.01, \mu=1.01(\text{LS}), \sigma_{\alpha}=0.3, \sigma_{\gamma}=0.01, \rho_n=-0.4, \rho_{\alpha}=-0.5, \rho_{\gamma}=-0.05, \rho_{\gamma_{\alpha}}=0.05. \)

Table 3 lists the results for the upper and lower bounds of American calls on dividend-paying stocks when \( r < q \). The parameters, \( \rho_n \) and \( \lambda \), exhibit the same relationship with exercise prices as shown in Table 2. However, this study finds that the upper bounds for this case are looser than for that where the dividend yield is smaller than the risk free rate. This result matches the earlier finding of Chung and Chang (2007), since the upper bounds perform best for American calls when the dividend rate is zero. Furthermore, the value of upper bounds for American calls with the CP process is more sensitive to parameter \( \lambda \) than for those with the VG and LS processes. Regarding the different Lévy processes in Tables 2 and 3, the CP processes have the largest theoretical values compared to the other two cases, whether the American call is out of the money, in the money or at the money. The above result might be the CP process only measuring the finite and large jump caused by special events for the individual company or market, but the jump results in excessive reactions to individual stock market events. Therefore, the stock price is compensated more values than the other two infinite activity jump cases. Finally, Table 2 and 3 list numerical results indicating that the VG and LS process have the second and smallest theoretical value.
Table 3. Our upper bounds for American calls on dividend-paying stocks when $r < q$

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>Processes</th>
<th>$K$</th>
<th>Processes</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>Upper CP</td>
<td>90</td>
<td>14.0196</td>
<td>13.1615</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>7.9985</td>
<td>7.3203</td>
</tr>
<tr>
<td></td>
<td></td>
<td>110</td>
<td>4.1325</td>
<td>3.6633</td>
</tr>
<tr>
<td></td>
<td>Lower VG</td>
<td></td>
<td>12.0916</td>
<td>11.3023</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6.6695</td>
<td>6.0657</td>
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<td></td>
<td></td>
<td>3.3246</td>
<td>2.9224</td>
</tr>
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<td>0</td>
<td>Upper CP</td>
<td>90</td>
<td>14.0221</td>
<td>13.5883</td>
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<tr>
<td></td>
<td></td>
<td>100</td>
<td>7.9962</td>
<td>7.6563</td>
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<td></td>
<td></td>
<td>110</td>
<td>4.1253</td>
<td>3.8944</td>
</tr>
<tr>
<td>0.4</td>
<td>Upper CP</td>
<td>90</td>
<td>14.0247</td>
<td>14.0196</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>7.9939</td>
<td>7.9985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>110</td>
<td>4.1181</td>
<td>4.1325</td>
</tr>
</tbody>
</table>

Note. The top and bottom numbers of every process are American and European call option values, respectively.

The parameters are $S(0)=100, v(0)=0.04, r(0)=0.05, q(0)=0.03, \alpha=1.5, \beta=1, \gamma_r=0.005, \gamma_q=0.003, k_r=0.03, k_q=0.04, \delta_r=0.05, \delta_q=0.05, \tau=1, \tau_q=1.5, \lambda=0.6, \mu=0.01, \mu=1.01 (LS), \sigma_s^2=0.3, \sigma_r=0.1, \rho_s=-0.4, \rho_r=-0.05, \rho_q=0.05, \rho_{\sigma_s}=0.05.$

Table 4 lists the upper bounds of American futures calls for four processes. Regardless of the process used, the differences between the upper and lower bounds are minimal. Moreover, Table 4 reveals that the upper and lower bounds of American futures calls exceed those of American calls when $r > q$. This result from the futures contract having longer maturity than the maturity of American call when $r > q$. Therefore, the numerical results presented here match expectations.
Table 4. Our upper bounds for American futures calls on dividend-paying stocks

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>Processes</th>
<th>$K$</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>$\lambda$</th>
<th>Processes</th>
<th>$K$</th>
<th>90</th>
<th>100</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>Upper</td>
<td>CP</td>
<td>16.3129</td>
<td>10.3394</td>
<td>6.1600</td>
<td>0.2</td>
<td>CP</td>
<td>15.4673</td>
<td>9.6435</td>
<td>5.6351</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lower</td>
<td>VG</td>
<td>15.8308</td>
<td>10.0338</td>
<td>5.9779</td>
<td></td>
<td>VG</td>
<td>15.0102</td>
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<tr>
<td></td>
<td></td>
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<td>15.7514</td>
<td>9.9408</td>
<td>5.8874</td>
<td></td>
<td>LS</td>
<td>15.2858</td>
<td>9.5159</td>
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</tr>
<tr>
<td>0</td>
<td>CP</td>
<td>16.3159</td>
<td>10.3376</td>
<td>6.1530</td>
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<td>CP</td>
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</tr>
<tr>
<td></td>
<td>VG</td>
<td>15.8337</td>
<td>10.0321</td>
<td>5.9711</td>
<td></td>
<td>VG</td>
<td>15.4184</td>
<td>9.6935</td>
<td>5.7203</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15.7556</td>
<td>9.9402</td>
<td>5.8814</td>
<td></td>
<td>LS</td>
<td>15.5193</td>
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<td>5.7172</td>
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</tr>
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<td>15.2899</td>
<td>9.6464</td>
<td>5.7075</td>
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<td>15.0606</td>
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</tr>
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<td>0.4</td>
<td>CP</td>
<td>16.3190</td>
<td>10.3359</td>
<td>6.1460</td>
<td>0.6</td>
<td>CP</td>
<td>16.3129</td>
<td>10.3394</td>
<td>6.1600</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>VG</td>
<td>15.8367</td>
<td>10.0304</td>
<td>5.9643</td>
<td></td>
<td>VG</td>
<td>15.8308</td>
<td>10.0338</td>
<td>5.9779</td>
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<td></td>
<td></td>
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<td>15.7597</td>
<td>9.9396</td>
<td>5.8754</td>
<td></td>
<td>LS</td>
<td>15.7514</td>
<td>9.9408</td>
<td>5.8874</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15.2940</td>
<td>9.6458</td>
<td>5.7017</td>
<td></td>
<td></td>
<td>15.2858</td>
<td>9.6470</td>
<td>5.7134</td>
<td></td>
</tr>
</tbody>
</table>

Note. The top and bottom numbers of every process are American and European call option values, respectively. The parameters are $S(0)=100, v(0)=0.04, r(0)=0.05, q(0)=0.03, \alpha=1.5, \beta=1, \gamma_2=0.005, \gamma_3=0.003, k_1=0.3, k_2=0.04, \delta_1=0.05, \delta_2=0.05, \tau_1=1, T_i=1.5, \lambda=0.6, \mu=0.01, \mu=1.01(\text{LS}), \sigma_1=0.3, \sigma_2=0.01, \rho_a=-0.4, \rho_v=-0.5, \rho_{sv}=0.05, \rho_{\sigma a}=0.05.$

6. Empirical analysis

The most popular approach to solving the inverse problem is to minimize the error or discrepancy between model prices and market prices, namely nonlinear least squares method. More specifically, the squared diversities between vanilla option market prices and model prices are minimized over the parameter space.

6.1 Estimation procedure

According to the estimation procedure of Bakshi, Cao and Chen (1997) and Huang and Wu (2004), the difference between market prices and model prices in Proposition 1 of $i$-th option is the function of value taken by $V(t)$ and by parameter vector $\Theta$ as follows:

$$
\epsilon_i[V(t), \Theta] = C^M_i(K_i, T_i) - C^\Theta_i(K_i, T_i)
$$

(27)

where $\Theta$ is a vector of parameter values, and $C^M_i(K_i, T_i)$ denote the market price of $i$-th option and $C^\Theta_i(K_i, T_i)$ represents its model price. Furthermore, $T_i$ and $K_i$ are respectively denoted by the time to expiration and the strike price of $i$-th option and $N$ is the number of options used for calibration.

By minimizing the sum of squared pricing errors (hereafter referred to as SSE) as follows, the vector of model parameters, $\Theta$, can be estimated by:
\[
SSE(t) = \min_{V(t), \Theta} \sum_{i=1}^{N} |E_i[V(t), \Theta]|^2 ,
\]
and \(MSE(t)\) denotes the mean squared pricing errors as:
\[
MSE(t) = \min_{V(t), \Theta} \frac{1}{N} \sum_{i=1}^{N} |E_i[V(t), \Theta]|^2
\]

The smaller sample averages of daily SSEs and MSEs for a model mean that the better a model fits the option prices. Moreover, the smaller SSEs further conjectures that the model is capable of capturing different cross-sectional properties of the option prices at different dates.

6.2 Data description

Again, based on the estimation procedure of Bakshi, Cao and Chen (1997), to compare the Black and Scholes’s (1972) model and the Carr and Wu (2004) model with our model, the model test is employed by the daily closing quotes from the S&P 500 index call options across different strikes in Datastream. In the model, the market observed parameters include the spot stock price \(S\), spot interest rate \(R_f\), strike prices \(K\) and time to expiration \(T\), which all can be compiled by the Datastream database.

Table 5 shows summary statistics of call option prices, spot prices, time to expiration and spot interest rates in October 2010. As mentioned before, this study employs the S&P 500 index call options daily closing quotes as spot prices and 30 day Treasury bill rate which refers to spot risk free rate. Moreover, a unit of the time to expiration is from 0.2137 to 0.1562 year. Our underlying asset prices, which are the S&P 500 index prices, exist within a narrow range between 1137.03 and 1184.71 in the period of time. The extent of moneyness, \(k = \ln(K/S)\), of our sample is between -0.2352 and 0.3398, which is equally distributed by deep in-the-money options to deep out-of-the-money options. Additionally, the whole range of call option quotes also extensively spreads from 0.05 to 351.1.

<table>
<thead>
<tr>
<th>Call</th>
<th>(S)</th>
<th>(T)</th>
<th>(R_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>116.1978</td>
<td>1169.0465</td>
<td>0.1802</td>
</tr>
<tr>
<td>Median</td>
<td>67.2250</td>
<td>1169.7700</td>
<td>0.1781</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.0500</td>
<td>1184.7100</td>
<td>0.1562</td>
</tr>
<tr>
<td>Maximum</td>
<td>351.1000</td>
<td>1137.0300</td>
<td>0.2137</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.6033</td>
<td>-0.8993</td>
<td>0.2923</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>-1.0522</td>
<td>0.3401</td>
<td>-1.2088</td>
</tr>
<tr>
<td>Number of obs.</td>
<td>196</td>
<td>196</td>
<td>196</td>
</tr>
</tbody>
</table>

Notes. Summary statistics of the call price (\(Call\)), spot price (\(S\)), time to maturity (\(T\)) and spot interest rates (\(R_f\)) from October 1 to 22, 2010.

6.3 Performance measures

The comparisons of different models are based on the sample properties of the daily sum squared pricing errors (SSE) and mean squared pricing errors (MSE) defined in Eq. (28) and Eq. (29) under the estimated model parameters. Therefore, our analysis is based on both the in-sample mean squared errors of the first 134 sample points and the out-of-sample mean squared errors of the last 62 sample points.
Table 6. Model parameter estimates and performance comparisons

Panel A. Model parameter estimates

<table>
<thead>
<tr>
<th>Case/Parameters</th>
<th>$\alpha \beta$</th>
<th>$\sigma_s$</th>
<th>$\sigma_v$</th>
<th>$\rho_{sv}$</th>
<th>$\delta_v$</th>
<th>$a_r$</th>
<th>$\rho_{sr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Proposed Model</td>
<td>0.4766</td>
<td>0.3455</td>
<td>1.1351</td>
<td>0.6204</td>
<td>-0.7214</td>
<td>0.4036</td>
<td>-0.2161</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case/Parameters</th>
<th>$\delta_s$</th>
<th>$a_q$</th>
<th>$\rho_q$</th>
<th>$v_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.3370</td>
<td>-0.2350</td>
<td>0.2300</td>
<td>0.0269</td>
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</table>

<table>
<thead>
<tr>
<th>Case/Parameters</th>
<th>$\kappa$</th>
<th>$\sigma_s$</th>
<th>$\sigma_v$</th>
<th>$\rho_{sv}$</th>
<th>$v_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carr and Wu Model (2004)</td>
<td>4.7841</td>
<td>0.2711</td>
<td>2.0000</td>
<td>0.8849</td>
<td>0.0591</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case/Parameters</th>
<th>$\sigma_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black and Scholes Model</td>
<td>0.1902</td>
</tr>
</tbody>
</table>

Panel B. Model performance comparisons

In-sample and out-of-the-sample comparison

<table>
<thead>
<tr>
<th>Case/Errors</th>
<th>SSE$_I$</th>
<th>SSE$_O$</th>
<th>MSE$_I$</th>
<th>MSE$_O$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Proposed Model</td>
<td>799.9961</td>
<td>298.5683</td>
<td>5.9701</td>
<td>4.8156</td>
</tr>
<tr>
<td>Black and Scholes Model</td>
<td>1416.3000</td>
<td>462.0500</td>
<td>10.5694</td>
<td>7.4524</td>
</tr>
</tbody>
</table>

Overall sample comparison

<table>
<thead>
<tr>
<th>Case/Errors</th>
<th>SSE</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Proposed Model</td>
<td>1098.5644</td>
<td>5.6049</td>
</tr>
<tr>
<td>Black and Scholes Model</td>
<td>1878.3500</td>
<td>9.5834</td>
</tr>
</tbody>
</table>

Notes: The overall sum of squared errors (SSE), the overall mean of squared errors (MSE), in-sample sum squared errors (SSE$_I$), in-sample mean squared errors (MSE$_I$), out-of-the-sample sum squared errors (SSE$_O$), and out-of-the-sample mean squared errors (MSE$_O$).

While pricing option models, one always confronts the difficulty that many parameters in models are unobservable, namely the following parameters in our model, such as the volatility of S&P 500 index returns $\sigma_s$, the volatility of index returns volatilities $\sigma_v$, and the correlation coefficient between index returns and index returns volatilities $\rho_{sv}$ and so on. By applying minimum least squares method, it could be easier to obtain these unobservable parameters from historical data. Panel A of Table 6 exhibits the estimates of parameters in our proposed model, Carr and Wu (2004) model and Black and Scholes model by using the sample data. Panel B of Table 6 shows the empirical performances of Black and Scholes model, Carr and Wu model and our proposed model. The SSE and MSE are used to measure the goodness-of-fit of three models. The overall SSEs are 1878.35 for the Black and Scholes model, 1173.6501 for Carr and Wu (2004) model and 1098.5644 for our model. To three overall SSEs, adding the stochastic interest and dividend rate parameters in the model indeed improves the performances of Black and Scholes and Carr and Wu models. The overall MSEs of three models report the same phenomenon as well. Furthermore, with the aim of further analysis of model performances, this study also proposes estimations of in-sample and out-of-the-sample pricing errors in Panel B of Table 6, which is adopted by Huang and Wu (2004). At the start, our investigation utilizes the first 134 sample points to obtain parameter estimates, in-sample sum squared errors (SSE$_I$) and in-sample mean squared errors (MSE$_I$). Next, this study use these parameters estimates in the models to acquire out-of-the-sample sum squared errors (SSE$_O$) and out-of-the-sample mean squared errors (MSE$_O$) of the last 62 sample points. In short, this investigation finds the
compelling evidences that no matter the SSE\textsubscript{t} or the MSE\textsubscript{t} of our model are less than those of Black and Scholes and Carr and Wu (2004) models. Noteworthy is the fact that the SSE\textsubscript{0} and MSE\textsubscript{0} in Panel B of Table 6 provide the similar evidences supporting that our model displays better performances than Black and Scholes and Carr and Wu (2004) models do. The proposed model with significantly smaller SSE\textsubscript{0} and MSE\textsubscript{0} has better forecasting ability than the other two. Actually, our proposed model is more suitable for the currency options pricing model because of the foreign exchange rates traded in the currency market will have more volatile changes than the interest rates or dividend rates.

7. Conclusions

This study continues extending the model of Carr and Wu (2004) model to establish a more generalized case involving stochastic interest rates specification under the HJM framework. The proposed model contains almost all risk factors from the option literature and overcomes the limitations of the Black and Scholes (1973) model. This study also complete the empirical analysis for Black and Scholes model, Carr and Wu (2004) model and our proposed model, demonstrating our model do improving the model performance. Furthermore, our study fills in the gap by establishing a connection between American options and the time-changed Lévy processes using a theoretical upper bounds technique. This proposed technique enables analytical solutions for pricing American options using the FFT methodology. Moreover, the proposed technique resolves the problem of American options requiring highly complex numerical methods considering multiple state variables.
Appendix A

Under the risk neutral measure $Q$, $P(t,T)/B(0,t)$ is a $Q$ martingale, then

$$d \left( \frac{P(t,T)}{B(0,t)} \right) = P(t,T) \cdot d \left( \frac{1}{B(0,t)} \right) + \frac{1}{B(0,t)} \cdot d \left( P(t,T) \right) + d \left( P(t,T) \cdot \frac{1}{B(0,t)} \right)$$

$$= \frac{P(t,T)}{B(0,t)} \left[ -\int_0^T \alpha(t,u) du + \frac{1}{2} \left( \int_0^T \sigma(t,u) du \right)^2 \right] dt - \int_0^T \sigma(t,u) du \, dW_r^P$$

$$= \frac{P(t,T)}{B(0,t)} \left[ -\int_0^T \alpha(t,u) du + \frac{1}{2} \left( \int_0^T \sigma(t,u) du \right)^2 \right] dt + dW_r^P$$

$$= \frac{P(t,T)}{B(0,t)} \left[ \int_0^T \sigma(t,u) du \right] dW_r^Q.$$  \hspace{1cm} (A1)

Based on Girsanov’s theorem, the processes $dW_r^Q(t)$ and $W_r^Q(t)$ are defined by

$$dW_r^Q(t) = dW_r^r(t) - \lambda_r(t) dt$$  \hspace{1cm} (A2)

where $\lambda_r(t) = \left[ -\int_0^T \alpha_r(t,u) du + \frac{1}{2} \left( \int_0^T \sigma_r(t,u) du \right)^2 \right] / \int_0^T \sigma_r(t,u) du$, and then by the Radon-Nikodym derivative of $P$ with respect to $Q$

$$\frac{dQ}{dP} = \exp \left[ -\frac{1}{2} \int_0^T \lambda^2(u) du + \int_0^T \lambda(u) dW_r^r(u) \right]$$  \hspace{1cm} (A3)

with

$$\int_0^T \alpha(t,u) du = \frac{1}{2} \left( \int_0^T \sigma(t,u) du \right)^2 - \lambda(t) \int_0^T \sigma(t,u) du$$

$$\alpha_r(t) = \sigma_r(t) \left[ \int_0^T \sigma_r(t,u) du - \lambda_r(t) \right].$$

Assume $S_q(t) \equiv S(t) \cdot B_q(0,t)$, then under the risk neutral measure $Q$, $dS_q(t)$ is given by

$$\frac{dS_q(t)}{S_q(t)} = \left[ \mu_q(t) + \delta_q \lambda(t) + \delta_q \sigma(t)^2 + \sigma_q \lambda(t) + q(t) \right] dt + \delta_q dW_q^q(t) + \delta_q dW_q^r(t)$$

$$+ \sigma_q dW_q^r(t) + \left( e^{q(t)} - 1 \right) d\pi_q^q(t) - E^Q \left[ e^{q(t)} - 1 \right] d\pi_q^r(t).$$  \hspace{1cm} (A4)

Define

$$Z_q(t) = \frac{S(t) \cdot B_q(0,t)}{B_q(0,t)} = \frac{S_q(t)}{B_q(0,t)}.$$
Then, under the risk neutral measure $Q$, $dZ_q(t)$ is given by
\[
\frac{dZ_q(t)}{Z_q(t)} = \left[ \mu_q(t) + \delta_q \lambda_q(t) + \delta_{s_q} \lambda_{s_q}(t) + \sigma_q \lambda_q(t) + q(t) - r(t) \right] dt + \delta_q dW^q(t) + \sigma_q dW^q(t) + \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) - E^q \left[ \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) \right],
\]
(A5)

$Z_q(t)$ is a $Q$ martingale, if
\[
\mu_q(t) = r(t) - q(t) - \delta_{s_q} \lambda_q(t) - \delta_{s_q} \lambda_{s_q}(t) - \sigma_q \lambda_q(t).
\]
(A6)

Furthermore, define the forward price of $S(t)$ using the dividend rate as
\[
S_q(t,T) \equiv S(t) \cdot P_q(t,T).
\]

Then under the risk neutral measure $Q$, constructed by $\{W^q_s, W^q_r, W^q_q, x^q, q^q_s\}$, where
\[
dW^q_s = dW^p_s - \lambda_q(t) dt, \quad dW^q_r = dW^p_r - \lambda_q(t) dt, \quad dW^q_q = dW^p_q - \lambda_q(t) dt, \quad x^q = x^p, \quad \pi^q_s = \pi^p_s,
\]
satisfies a martingale condition and following dynamic as
\[
dS_q(t,T) = S(t) \cdot dP_q(t,T) + dS(t) \cdot P_q(t,T) + d \left\{ S(t) \cdot P_q(t,T) \right\}
\]
\[
= S(t) \cdot P_q(t,T) \left[ q(t) - \int_0^T \sigma_q(t,u) du + \frac{1}{2} \left( \int_0^T \sigma_q(t,u) du \right)^2 \right] dt
\]
\[
- \int_0^T \sigma_q(t,u) du \left( dW^q(t) + \lambda_q(t) \right) + S(t) \cdot P_q(t,T) \left[ (r(t) - q(t)) dt \right]
\]
\[
+ \delta_q dW^q(t) + \delta_{s_q} dW^q(t) + \sigma_q dW^q(t) + \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) - E^q \left[ \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) \right]
\]
\[
= S(t) \cdot P_q(t,T) \left[ (r(t) - \int_0^T \sigma_q(t,u) du + \frac{1}{2} \left( \int_0^T \sigma_q(t,u) du \right)^2 \right] dt
\]
\[
- \int_0^T \sigma_q(t,u) du \lambda_q(t) \left[ \left( \delta_{s_q} \rho_q + \delta_{s_q} + \rho_q \sigma_q \right) \cdot \left( \int_0^T \sigma_q(t,u) du \right) \right] dt
\]
\[
+ S(t) \cdot P_q(t,T) \left[ \delta_{s_q} dW^q(t) + \delta_{s_q} dW^q(t) + \sigma_s dW^q(t) \right.
\]
\[
+ \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) - E^q \left[ \left( e^{\rho_q(t)} - 1 \right) d\pi^q(t) \right].
\]
(A7)

Define
\[
Z_{q(t)} = \frac{S(t) \cdot P_q(t,T)}{B_q(0,t)}.
\]
(A8)

Then, under the risk neutral measure $Q$, $dZ_{q(t)}$ is given by

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\[
\frac{dZ_{S_{q}(t)}}{Z_{S_{q}(t)}} = \left[ -\int_{0}^{T} \alpha_{q}(t,u)du + \frac{1}{2} \left( \int_{0}^{T} \sigma_{q}(t,u)du \right)^2 dt - \int_{0}^{T} \sigma_{q}(t,u)du \lambda_{q}(t) \right] \\
+ \left[ \left( \delta_{S_{q},\rho_{q}} + \delta_{S_{q},\rho_{q}} \right) - \int_{0}^{T} \sigma_{q}(t,u)du \right] dt + \delta_{S_{q},dW^{Q}_{S_{q}}}(t) + \delta_{S_{q},dW^{Q}_{q}}(t) \\
+ \sigma_{S_{q}}dW^{Q}_{S_{q}}(t) + \left( e^{\psi_{q}(t)} - 1 \right) d\pi^{Q}_{S_{q}}(t) - E^{Q} \left[ \left( e^{\psi_{q}(t)} - 1 \right) d\pi^{Q}_{S_{q}}(t) \right].
\]  

(A9)

Moreover, \( Z_{S_{q}(t)} \) is a martingale if
\[
\int_{t}^{T} \alpha_{q}(t,u)du = \left[ -\int_{0}^{T} \alpha_{q}(t,u)du + \frac{1}{2} \left( \int_{0}^{T} \sigma_{q}(t,u)du \right)^2 dt - \int_{0}^{T} \sigma_{q}(t,u)du \lambda_{q}(t) \right] \\
+ \left[ \left( \delta_{S_{q},\rho_{q}} + \delta_{S_{q},\rho_{q}} \right) - \int_{0}^{T} \sigma_{q}(t,u)du \right] 
\]  

(A10)

and under the risk neutral measure \( Q \), the dynamic of the zero coupon bond for dividend rate
\[
\alpha_{q}(t,T) = \sigma_{q}(t,T) \left[ \left( \int_{0}^{T} \sigma_{q}(t,u)du \right) dt - \lambda_{q}(t) - \delta_{S_{q},\rho_{q}} - \delta_{S_{q},\sigma_{S}} \right].
\]  

(A11)

Appendix B

The following proves Proposition 3. The first step is defining the forward price of \( S(t) \) based on the dividend rate
\[
S_{v}(t,T) \equiv S(t) \cdot P_{q}(t,T).
\]  

(B1)

Then by Itô’s lemma, the dynamic of \( S_{v}(t,T) \) is
\[
dS_{v}(t,T) = S(t) \cdot dP_{q}(t,T) + dS(t) \cdot P_{q}(t,T) + d \left( S(t) \cdot P_{q}(t,T) \right) \\
= S(t) \cdot P_{q}(t,T) \left[ \left( q(t) - \left( \delta_{S_{q},\rho_{q}} + \delta_{S_{q},\sigma_{S}} \right)a_{q}(t,T)dt + a_{q}(t,T)dW^{Q}_{q}(t) \right) \\
+ S(t) \cdot P_{q}(t,T) \left[ \left( r(t) - q(t) \right) dt + \delta_{S_{q},dW^{Q}_{S_{q}}}(t) + \delta_{S_{q},dW^{Q}_{q}}(t) + \sigma_{S_{q}}dW^{Q}_{S_{q}}(t) \right) \\
= S(t) \cdot P_{q}(t,T) \left[ r(t)dt + \delta_{S_{q},dW^{Q}_{S_{q}}}(t) + \delta_{S_{q},dW^{Q}_{q}}(t) + \sigma_{S_{q}}dW^{Q}_{S_{q}}(t) \right] \\
+ \left( e^{\psi_{q}(t)} - 1 \right) d\pi^{Q}_{S_{q}}(t) - E^{Q} \left[ \left( e^{\psi_{q}(t)} - 1 \right) d\pi^{Q}_{S_{q}}(t) \right].
\]  

(B2)
and using Itô’s lemma, define $S^\gamma_v(t,T_t) = \frac{S_v(t,T_t)}{P(t,T_t)}$

$$d\left(S^\gamma_v(t,T_t)\right) = S_v(t,T_t) \cdot d\left(\frac{1}{P(t,T_t)}\right) + dS_v(t,T_t) \cdot \frac{1}{P(t,T_t)} + d\left(S_v(t,T_t) \cdot \frac{1}{P(t,T_t)}\right)$$

$$= S^\gamma_v(t,T_t) \left(\left(-r(t) + a_v(t,T_t)^2\right) + a_v(t,T_t)dW^\gamma_v(t)\right)$$

$$+ S^\gamma_v(t,T_t) \left[r(t)dt + \delta_s dW_s^Q(t) + \left(\delta_q + a_q(t,T_t)\right)dW_q^Q(t)\right]$$

$$+ \sigma_s dW_s^Q(t) + \left(e^{\gamma(t)} - 1\right)d\pi_s^Q(t) - E^Q \left[\left(e^{\gamma(t)} - 1\right)d\pi_s^Q(t)\right]$$

$$+ S^\gamma_v(t,T_t) \left[-\delta_s a(t,T_t)dt - \sigma_s \rho_s a(t,T_t)dt - \delta_q \rho_q a(t,T_t)dt\right]$$

$$= S^\gamma_v(t,T_t) \left[a_v(t,T_t)^2 - \delta_s a_v(t,T_t) - \sigma_s \rho_s a_v(t,T_t) - \delta_q \rho_q a_v(t,T_t)\right]dt$$

$$+ S^\gamma_v(t,T_t) \left(\delta_s - a_v(t,T_t)\right)dW^\gamma_v(t) + \left(\delta_q + a_q(t,T_t)\right)dW^\gamma_q(t)$$

$$+ \sigma_s dW_s^Q(t) + \left(e^{\gamma(t)} - 1\right)d\pi_s^Q(t) - E^Q \left[\left(e^{\gamma(t)} - 1\right)d\pi_s^Q(t)\right].$$

(B3)

Then use the forward neutral measure to define the relative price of a zero coupon bond for interest rate, $P^r_v(t,T_t) = \frac{P(t,T_t)}{P(t,T)}$, in which case $d\left(P^r_v(t,T_t)\right)$ is given by

$$d\left(\frac{P^r_v(t,T_t)}{P^\gamma_v(t,T_t)}\right) = a_v(t,T_t) \left(a_v(t,T_t) - a_v(t,T_t)\right)dt + \left(a_v(t,T_t) - a_v(t,T_t)\right)dW^\gamma_v(t).$$

(B4)

Using Girsanov’s theorem, the processes $W^\gamma_v(t)$ and $W^\gamma_q(t)$ are defined by

$$dW^\gamma_v(t) = dW^\gamma_v(t) - \lambda_v(t)dt$$

$$= dW^\gamma_v(t) - a_v(t,T_t)dt$$

(B5)

where $\lambda_v(t) = \frac{a_v(t,T_t) - a_v(t,T_t)}{a_v(T_t) - a_v(T_t)}$, and furthermore

$$d\left(\frac{P^r_v(t,T_t)}{P^\gamma_v(t,T_t)}\right) = \left(a_v(t,T_t) - a_v(t,T_t)\right)dW^\gamma_v(t).$$

(B6)

Substituting $dW^\gamma_s = dW^\gamma_s - a_v(t,T_t)\rho_s dt$, $dW^\gamma_q = dW^\gamma_q - a_v(t,T_t)dt$,

$$dW^\gamma_s = dW^\gamma_s - a_v(t,T_t)\rho_s dt, \quad x^\gamma = x^0, \quad \text{and} \quad \pi^\gamma_s = \pi^0_s$$

into Eq.(B2) yields

$$\frac{dS^\gamma_v(t,T_t)}{S^\gamma_v(t,T_t)} = \left[\delta_s - a_v(t,T_t)\right]dW^\gamma_v(t) + \left[\delta_q + a_q(t,T_t)\right]dW^\gamma_q(t) + \sigma_s dW^\gamma_v(t)$$

$$+ \left(e^{\gamma(t)} - 1\right)d\pi^\gamma_v(t) - E^Q \left[\left(e^{\gamma(t)} - 1\right)d\pi^\gamma_v(t)\right].$$

(B7)
This completes the proof of Proposition 3, and then the characteristic exponent of diffusion of \( S_T^T(t,T) \) under the forward-neutral measure is obtained as

\[
\psi_a = \frac{1}{2} \sigma^2 (iu + u^2)T,
\]

where \( T \) represents random time changes between 0 to \( t \) and

\[
\sigma^2 = \left[ \delta_{s_r} - a_r(t,T) \right]^2 + \left[ \delta_{s_q} + a_q(t,T) \right]^2 + \sigma_s^2 + 2 \rho_{s_r} \left( \delta_{s_r} - a_r(t,T) \right) \left( \delta_{s_q} + a_q(t,T) \right) \\
+ 2 \rho_{s_q} \left( \delta_{s_r} - a_r(t,T) \right) \sigma_S + 2 \rho_{s_q} \left( \delta_{s_q} + a_q(t,T) \right) \sigma_S.
\]

**Appendix C**

The futures price must equal the spot price at maturity \( T \), and thus the futures price at time \( t \) is

\[
F(t,T) = E \left[ F(T,T) \mid \mathcal{F}_t \right] = E \left[ S(T) \mid \mathcal{F}_t \right] \\
= S(t)e^{\int_t^{T} (f(t,s) - f(t,s))ds} \\
= \frac{S(t)P_q(t,T)}{P_r(t,T)}.
\]

In terms of Eq.(B3) in Appendix B, the futures price process is given by

\[
dF(t,T) = F(t,T) \exp \left[ \left( a_r(t,T)^2 - \delta_{s_r} a_r(t,T) - \sigma_s \rho_{s_r} a_r(t,T) - \delta_{s_q} \rho_{s_q} a_r(t,T) \right) dt \\
+ \left( \delta_{s_r} - a_r(t,T) \right) dW_r^Q(t) + \left( \delta_{s_q} + a_q(t,T) \right) dW_q^Q(t) \\
+ \sigma_s dW_S^Q(t) + \left( e^{\psi(t)} - 1 \right) d\pi_S^Q(t) - E \left[ \left( e^{\psi(t)} - 1 \right) d\pi_S^Q(t) \right].
\]

Using the martingale property, the price of European call options with maturity date \( T \) and exercise price \( K \) on the futures contract with maturity \( \tau \) is calculated by

\[
C^F = E_0^Q \left[ \frac{1}{B_r(0,T)} (F(T,T_t) - K)^+ \mid \mathcal{F}_0 \right] \\
= P_r(0,T) E_0^Q \left[ \frac{1}{P_r(T,T_t)} (F(T,T_t) - K)^+ \mid \mathcal{F}_0 \right] \\
= P_r(0,T) E_0^Q \left[ (F(T,T_t) - K)^+ \mid \mathcal{F}_0 \right].
\]
\[
\begin{align*}
&d \left( \frac{F(T,T_t)}{P_r(T,T)} \right) = F(0, T_t) \exp \left[ (\delta_{S_T}(t) - a_r(t,T_t) - a_s(T,T))dW^{Q_T}_r(t) + (\delta_{S_q}(t) + a_q(t,T_t))dW^{Q_T}_q(t) \right] \\
&\quad + \sigma_d W^{Q_T}_S(t) + \left( e^{\varphi(t)} - 1 \right) d\pi^{Q_T}_S(t) - E^{Q_T} \left[ \left( e^{\varphi(t)} - 1 \right) d\pi^{Q_T}_S(t) \right] \quad (C4)
\end{align*}
\]
where
\[
\begin{align*}
&dW^{Q_T}_S = dW^Q_S - \lambda_S(t) dt, \quad dW^{Q_T}_r = dW^Q_r dt, \quad dW^{Q_T}_q = dW^Q_q dt, \quad \lambda^Q(t) = \frac{1}{\sigma^Q}(r(t) + a_r(t,T_t)^2 - \delta_{S_T}a_s(T,T_t) - \sigma_S \rho_{S_S} a_T(T,T_t) - \delta_{S_q} \rho_{S_q} a_q(T,T_t)) \\
&\quad - a_r(t,T_t) \left( \delta_{S_T} - a_r(T,T_t) \right) - a_s(T,T_t) \left( \delta_{S_q} + a_q(T,T_t) \right) \rho_{S_S} - a_s(T,T_t) \sigma_S \rho_{S_S} dt,
\end{align*}
\]
and then we can obtain the characteristic exponent of diffusion of \( F(t,T_t)/P_r(t,T) \) can be obtained under the forward-neutral measure as
\[
\psi_s = \frac{1}{2} \sigma^2 (iu + u^2) T,
\]
where \( T_r \) as the random time changes from 0 to \( t \) and
\[
\sigma^2 = \left[ \delta_{S_T} - a_r(T,T_t) - a_s(T,T) \right]^2 + \left[ \delta_{S_q} + a_q(T,T_t) \right]^2 + \sigma_S^2 + 2 \rho_{S_S} \left( \delta_{S_T} - a_r(T,T_t) - a_s(T,T) \right) \\
\times \left( \delta_{S_q} + a_q(T,T_t) \right) + 2 \rho_{S_S} \left( \delta_{S_T} - a_r(T,T_t) - a_s(T,T) \right) \sigma_S + 2 \rho_{S_S} \left( \delta_{S_q} + a_q(T,T_t) \right) \sigma_S.
\]

References